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ELECTROMAGNETIC FIELDS INDUCED BY OCEAN CURRENTS

Wasyl Wasylkiwskyj

July 1979

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ABSTRACT

This paper provides a comprehensive account of analytical results for computing electromagnetic fields that are induced by ocean sea water as a result of its motion relative to the geomagnetic field. The emphasis is on the characterization of magnetic field and magnetic field gradient spectra induced by internal waves and surface waves in a deep ocean environment. The theoretical results are formulated so as to be directly applicable to the computation of sea water generated magnetic noise and to the assessment of its deleterious effects on the sensitivity of magnetic sensors employed for magnetic anomaly detection over an open ocean. Magnetic field component and gradient spectra are computed both for stationary and moving sensor observation platforms.

Mathematics may be compared to a mill of exquisite workmanship
which grinds you stuff of any degree of fineness,
but nevertheless what you get out depends on what you put in --
and as the grandest mill in the world
will not extract wheat flour from peapods,
pages of formulae will not get a definite result out of loose data.

. . . T.H. Huxley

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SUMMARY AND CONCLUSIONS

This paper comprises results of an analysis of electromagnetic fields induced by the interaction of ocean currents with the geomagnetic field. The work is part of an ongoing effort at IDA in support of the DARPA program on nonacoustic ASW techniques. The objective of the effort reported on herein was to develop a general analytical formulation for the computation of electromagnetic field spectra induced by the motion of sea water in the upper layers of a deep ocean with particular emphasis on surface waves and internal waves.

The intended application of the analytical and numerical results is to the characterization of ocean current generated magnetic noise that could degrade the performance of sensitive instruments (e.g., superconducting gradiometers) employed in magnetic anomaly detection over a deep ocean. Although existing instruments that respond directly to ocean wave generated electromagnetic noise are predominantly of the magnetic type (measuring induced magnetic fields or their gradients), in this study electric fields are also under consideration. The purpose of including electric fields within the same analytical framework is twofold. First, the inclusion of the electric field elucidates the physical mechanisms responsible for the interaction between the geomagnetic field and hydrodynamic phenomena in the ocean. Second, under certain conditions, the electric field comprises information on the hydrodynamic flow field not readily inferred from magnetic-type measurements alone.

A brief outline of the material covered in this paper is as follows:

1. The derivation of expressions for electromagnetic fields induced by general fluid velocity fields, followed by an investigation of the validity of various approximations to the field equations. The discussion includes a comparative evaluation of approximations employed in past studies of magnetohydrodynamic phenomena (Chapters I-III, together with Appendices B, C, and D).
2. Explicit expressions for components of induced magnetic fields under the quasi-static approximation but arbitrary fluid velocity fields (Chapter IV).
3. Analytical results for electromagnetic fields induced by linear surface waves, and linear internal waves in a deep ocean (Chapter V).
4. Explicit expressions and numerical results for the spatial and temporal spectra of surface-wave- and internal-wave-induced electromagnetic field components and their gradients observed from stationary (Chapter V) and moving (Chapter VI) measurement platforms.

The analytical formulation for surface-wave-induced and internal-wave-induced electromagnetic field spectra requires several fundamental assumptions on hydrodynamic phenomena in the ocean. The required background material is presented in Appendix A for surface waves and in Appendix E for internal waves.

The major conclusions are:

- The quasi-static approximation to the electromagnetic fields is valid if the horizontal scale length of the hydrodynamic flow fields does not exceed 1 km, and if the characteristic frequency is on the order of 1 Hz or less. This encompasses the usual range of hydrodynamic phenomena of interest in magnetic anomaly detection. Under the quasi-static approximation the electromagnetic fields are given by relations derived from electrostatics and magnetostatics, wherein time enters only as a

parameter. This approximation affords a substantial simplification and permits a unified treatment of electromagnetic fields induced by ocean currents.

- Under quasi-static approximation internal waves induce magnetic fields above the ocean surface, *but no electric fields*. On the other hand, surface waves induce both magnetic and electric fields. Moreover, the functional forms of the temporal spectrum of any component of the surface-wave-induced electric field and a surface-wave-induced magnetic field gradient are identical. This feature could be exploited in subtracting the surface-wave-induced contribution from the internal wave contribution in a moving gradiometer sensor. Thus, since for sufficiently fast platform velocities the surface wave and internal wave contributions to a measured magnetic field gradient overlap (see, e.g., Fig. 16a, p. 143), their separation on the basis of a total (spectral) power measurement would not be possible. An electric field sensor would provide an independent measurement of the surface wave contribution, which could be subtracted from the total gradiometer output (e.g., by employing a correlation technique).
- Numerical results based on the theory developed for the spectra of magnetic field gradients induced by surface waves and internal waves indicate levels substantially above the intrinsic instrument noise limit of currently available superconductive gradiometers. For example, for an aircraft-mounted gradiometer typical computed spectra are shown in Fig. 16a, p. 143. Over the frequency range shown, the intrinsic noise level of the instrument would be essentially flat at 10^{-3} $(pT/m)^2/Hz$ for a state-of-the-art device and at about 10^{-2} $(pT/m)^2/Hz$ for an "average" gradiometer sensor.

- Based on the theory developed herein, the temporal spectra of the internal-wave-induced magnetic field gradients observed from a uniformly translating measurement platform above the ocean surface are relatively insensitive to the detailed structure of the thermocline, *provided* observations are restricted to the frequency range above the maximum Väisälä frequency. This result holds true only if the platform velocity exceeds the maximum internal wave group velocity (typically a fraction of a meter per second).
- Horizontal and vertical components of the internal-wave-induced magnetic field components and gradients observed from a geostationary measurement platform are completely decorrelated whenever the internal wave wavenumber spectrum is isotropic. Conversely, the degree of correlation between such components is a measure of the directionality of the internal wave wavenumber spectrum. Thus, correlation techniques applied to orthogonal components of the induced magnetic field gradient afford the possibility of more accurate determination of internal wave spectrum directionality than currently possible from direct hydrodynamic measurements.
- A single-axis magnetic field gradient sensor has also modest intrinsic directional discrimination properties. Depending on the relative orientation of the sensor axis and the geomagnetic field, the "gain" in detecting a perfectly directional internal wave field relative to an isotropic background internal wave of equal power can reach about 6.8 dB.

CONTENTS

I. INTRODUCTION

This paper provides a comprehensive account of analytical results for computing electromagnetic fields that are induced by ocean sea water as a result of its motion relative to the geomagnetic field. There has been a sustained interest in this area over a number of years mainly due to the potential application of this class of phenomena as a diagnostic tool in oceanography. More recently interest has been generated by problems in magnetic anomaly detection over an open ocean. In this class of problems, ocean current generated magnetic fields and their gradients constitute a source of noise.

The majority of past analytical studies deal with a restrictive class of hydrodynamic flows. Thus, Longuet-Higgins et al. [1] treat electric fields induced by steady motion of sea water. The papers of Worburton and Cominiti [2] and Weaver [3] consider surface-wave-induced electromagnetic fields. Internal-wave-induced magnetic fields for a two-layer ocean model are treated by Beal and Weaver [4], employing the formulation for irrotational velocity fields in [3]. Sanford [5] treats electromagnetic fields generated by deep-sea tides. Employing the stochastic Pierson-Neumann spectrum model for wind-generated surface waves, Bergin [6] has presented calculations of average magnetic fields induced in a deep ocean. The most comprehensive treatment of ocean-wave-induced electromagnetic fields is due to Podney [7]. It encompasses surface waves and internal waves for oceans with arbitrary horizontal stratification.

No formulation applicable to general oceanic flow fields appears to have been published. Thus, although Podney's [7]

results encompass rotational flow, they are confined to the special case of purely horizontal vorticity. In addition, in the existing literature, the treatments of the permissible approximations to the electromagnetic field equations are generally specialized to the particular flow field under discussion, so that it is not always clear whether and under what conditions the approximations may be extended to encompass more general situations. Moreover, statements with regard to the range of validity of various approximations are by no means consistent. For example, according to Sanford [5] the quasi-static approximation is valid if $\sigma\mu_o\omega H\lambda < 1$, where H is the ocean depth and λ the horizontal scale length of the hydrodynamic wave, clearly implying that such an approximation breaks down for a sufficiently deep ocean. On the other hand, according to Podney [7], the restriction on the validity of the quasi-static approximation is of the form $\sigma\mu_o\omega\lambda^2 << 1$, which, in consequence, appears applicable to an ocean of arbitrary depth. Other approximations whose nature is not clarified are implicit in the existing formulations. A case in point is the electric field above the ocean surface induced by surface waves. Thus, one finds that this electric field does not vanish even in the limit of zero electric conductivity of the fluid [see, e.g., Eq. (27d) of Podney [7]]. Such a result is clearly inadmissible for it would mean generation of an electric field by an "ether wind."

The primary motivation for the work presented herein was to construct theoretical models for temporal and spatial spectra of internal-wave and surface-wave-induced magnetic fields and their gradients, the results forming the basis for further study of the effects of these noise sources on the sensitivity of instruments employed for magnetic anomaly detection over a deep ocean. However, because of the apparent lack of generality in, and the perceived inconsistencies of, the formulations in the published literature, it was deemed advisable to reformulate the problem by starting from first principles, so as to encompass arbitrary

flow fields and, at the same time, carefully examine the quantitative significance of the required approximations.

The formulation for electromagnetic fields induced by arbitrary oceanic currents is developed in Chapters II, III, and IV, in conjunction with Appendixes B, C, and D. It is shown by simple arguments presented in Chapter III and by a rigorous analysis in Appendix D that the restriction on the quasi-static approximation is expressed by the inequality $\lambda << 10^3 f^{-1/2}$, wherein λ is the horizontal scale of the hydrodynamic disturbance in meters and f the frequency in Hz. This agrees with the condition given by Podney [7] but is in disagreement with that of Sanford [5]. Specifically, under the quasi-static approximation, the time-varying electromagnetic fields induced by a velocity field $\underline{V}(\underline{r},t)$ are identical to those obtained from the solution of purely magnetostatic and electrostatic problems except that the time variable appears explicitly as a parameter in the forcing functions (velocity fields). In anticipation of this result the discussion in Chapter II deals exclusively with electrostatics and magnetostatics. The electrostatic problem, set up in its full generality, at once reveals the reason for the apparent lack of dependence of the electric field above the ocean surface on conductivity. It turns out that this independence is only approximate, since it is valid under the stipulation that $\varepsilon_o \varepsilon_r / \sigma << 1$ sec, where ε_r is the dielectric constant of the fluid. This approximation is fully justified for sea water and, therefore, makes it perfectly clear why the limiting form for zero conductivity of Podney's [7] Eq. 27d is not meaningful.

The connection with Podney's [7] formulation for the electric field in terms of the vector stream function is made in the discussion on pages 23-29, (specifically Eq. 59b), where is is shown that a term must be added to Podney's [7] result when the vertical vorticity of the fluid is not identically zero. The presence of this additional term renders the formulation for the

magnetic field somewhat more cumbersome, even under the quasi-static approximation. The explicit results for all the magnetic field components, (both below and above the ocean surface) are listed in Chapter IV. The formulae are given in two equivalent forms: as volume integrals over the velocity fields, and in terms of the Fourier transforms of the velocity fields. Depending on the manner in which the velocity fields are prescribed, one of the forms may be more convenient.

In Chapter V the results are specialized to electromagnetic fields induced by internal waves and surface waves. Under the quasi-static approximation the formulas agree exactly with those presented by Podney [7]. Although the quasi-static approximation within the stipulated restriction gives an adequate account of the dominant field components, certain characteristic physical features of the structure of the induced electromagnetic fields emerge only when a full wave solution is considered. The detailed analysis is presented in Appendix D, and the results are discussed in Chapter V-D. One finds that, when viewed in light of transport of electromagnetic energy above the ocean surface, internal-wave-induced fields differ from surface-wave-induced fields in rather fundamental respects. Thus, a unidirectional internal wave gives rise to an electromagnetic surface wave above the ocean surface. The direction of propagation, the group velocity, and phase velocity of this electromagnetic surface wave are identical to those of the internal wave. This electromagnetic surface wave is of the H-mode type: the vertical magnetic field and the electric field parallel to the ocean surface and orthogonal to the direction of propagation, form the electromagnetic pair whose product gives rise to a real component of the Poynting vector in the direction of propagation; the third field component is a magnetic field that points in the direction of propagation (hence, the designation H-mode). The direction of real electromagnetic power flow is thus always perpendicular to the wave crest of the internal wave. The situation is fundamentally different for surface-wave-induced electromagnetic

fields. One finds that in this case one obtains two electromagnetic surface waves: an H-mode wave, and an E-mode wave, each when taken in isolation, carries electromagnetic power perpendicular to the wave crest. The structure of the E-mode wave is characterized by a vertical electric field, and a magnetic field parallel to the ocean surface and perpendicular to the direction of propagation. The third component is that of the electric field which points in the direction of propagation. Podney [7] refers to the electric field associated with this wave as an electrostatic field, a designation which is misleading since this field obviously depends on time and participates in transport of real electromagnetic power. Since, in general, a hydrodynamic surface wave induces both an H-mode and an E-mode wave, coupling between the two electromagnetic wave types gives rise to net electromagnetic power flow which is nearly along the crest of the inducing surface wave.

The relationship among the field components for each of the two electromagnetic surface waves turns out to be identical to that for classic electromagnetic slow surface waves. In particular, they could be generated by a process of total internal reflection of an electromagnetic plane wave impinging from within a dielectric half space on an air dielectric boundary. Of course, the value of the equivalent dielectric constant required for a simulation of the low phase velocities of these waves above the ocean surface would have to be extremely large ($\sim 10^7$).

Chapter VI takes up the statistical formulation for the electromagnetic fields induced by internal waves and surface waves. Under the assumption of temporal stationarity and spatial homogeneity in any horizontal plane, general formulas are derived for the spectra of electric fields, magnetic fields and their gradients as these would be observed from stationary platforms above the ocean surface. Internal wave induced fields are treated in Chapter VI-A, B. Under the quasi-static approximation, internal waves induce only magnetic fields above the ocean

surface. A complete characterization of the magnetic field and gradient spectra requires knowledge of the ocean stratification and the distribution of energy in mode wave number space. A theoretical model for the spectra of deep-ocean internal waves has been presented by Garrett and Munk [8]. We have found, however, that their model is not directly usable in computing spectra of induced magnetic fields. A different model, due to Milder [9], which incorporates some features of the Garrett and Munk model as a special case, was found more suitable for our purpose. In order to clarify the nomenclature employed in connection with the spectral calculations, a detailed account of the theory of linear internal waves is presented in Appendix E, which includes a comparison of the theories of Garrett and Munk and Milder. The central assumption which we employ throughout in our calculation of spectra of internal waves is that the energy in mode wave number space of ambient internal waves is distributed in proportion to the square of the phase velocities of the individual internal wave modes. We term this the Milder hypothesis. Its consequences are explored in detail in Appendix E, in particular in regard to the simplification it introduces in the expressions for "towed" internal wave spectra. Milder's hypothesis is incorporated into the formulation of internal-wave-induced magnetic field and gradient spectra. One important consequence of the hypothesis is that the spatial spectra of the induced fields can be computed directly from the knowledge of the Väisälä frequency profile without the need of computing the eigenfunctions and the associated dispersion relations. The results are applied to compute the average values of fields and gradients for a deep ocean. Although an exponentially decreasing Väisälä frequency profile has been used in these calculations, similar results can easily be obtained for arbitrary profiles, since the formulas are expressed explicitly in terms of the Väisälä frequency. On the other hand, the computation of temporal spectra requires a detailed knowledge of the internal

wave eigenfunction. Numerical results have been obtained for the spectra of magnetic field components and gradients for the exponentially decreasing Väisälä frequency profile. All data have been presented in a normalized form so that numerical values of the spectra can be obtained for arbitrary relative orientation of the geostationary coordinate system and the direction of the geomagnetic field. Although these numerical results have been obtained specifically for an isotropic internal wave spectrum, the general formulas are valid for arbitrary internal wave number directionality. The question of the feasibility of discriminating between an isotropic and highly directional internal wave spectra by means of multiple axes magnetic sensors is explored. In principle, such discrimination appears possible either on the basis of a spectral correlation measurement, or by taking advantage of the intrinsic directionality of the magnetic field component or gradient sensor. For example, one finds that the intrinsic directional discrimination of a single axis (horizontal-horizontal) gradient sensor is about 6.8 dB.

Surface-wave-induced electromagnetic field spectra are discussed in Chapter VI-C,D,E. Numerical calculations are based entirely on the Pierson-Neumann spectrum. Results for the total r.m.s. magnetic field agree with those presented by Bergin [6]. Numerical results are also obtained for magnetic field gradients as well as for the components of the induced electric field.

One interesting result provided by the analysis in Chapter VI-D,E is that the functional dependence on frequency of the magnetic field gradient spectrum and the electric field component spectrum is identical. Since internal waves induce no electric field above the ocean surface, the measurement of the mutual spectral coherence function of the electric field and of the magnetic field gradient could provide a means of identifying the surface wave spectral contribution in the output of a

magnetic field gradient sensor. A typical level of the total r.m.s. electric field at the ocean surface is on the order of 60 μvolts/meter.

Magnetic field spectra relative to a moving measurement platform are discussed in Chapter VII. In Chapter VII-A numerical results are presented for surface-wave-induced magnetic field and gradient spectra as would be observed from a low-flying aircraft above the ocean surface. The analytical results for computing the temporal spectra of internal-wave-induced magnetic fields and gradients are presented in Chapter VII-B. For tow speeds much greater than the maximum internal wave group velocities and temporal frequencies above the maximum Väisälä frequency, the formulae for internal-wave-induced magnetic field and gradient spectra can be expressed in a particularly simple form, viz., the spectra are given explicitly in terms of the Väisälä frequency profile. Thus, although numerical results have been obtained only for the exponential profile, similar calculations could easily have been carried out for any prescribed profile.

II. STATIC ELECTRIC AND MAGNETIC FIELDS INDUCED BY STEADY FLOW OF CONDUCTING FLUID THROUGH A CONSTANT MAGNETIC FIELD

A. FIELD EQUATIONS FOR A MOVING MEDIUM

In working toward the objective of establishing a general set of explicit relationships between the hydrodynamic velocity field in the ocean and the electromagnetic fields induced by the motion of the conducting sea water relative to the geomagnetic field, we shall start with the purely static situation, i.e., we shall temporarily assume that the water motion is steady. The transition to time-varying fields engendered by the usually unsteady flow will be made only in Chapter III. This approach is taken primarily for didactic reasons. The approximations to the electromagnetic field equations when specialized to the constitutive parameters of sea water are more readily arrived at for the static case. Subsequently it will be shown that for the normal range of temporal variations encompassed by ocean wave phenomena, and for characteristic spatial scales of much less than 1 km, the dominant components of the time varying fields are given by the same expressions as are the static fields provided one includes time as an additional parameter in the source terms (hydrodynamic velocity fields).

We shall assume throughout that the ocean surface is perfectly planar with the cartesian coordinate system oriented such that y is the local vertical, $y \geq 0$ defining the region above the ocean surface. In most of the discussion (the exception being Appendix D), the effects of the ocean bottom will be ignored, since we are primarily interested in formulating problems for

the case of a deep ocean. Thus, for the purpose of analysis, we take $-\infty < y \leq 0$ as the region occupied by sea water. Rationalized MKS units will be employed throughout.

Let $\underline{B}_o$ = earth's magnetic field which is taken as constant, $\underline{V}$ the velocity of the fluid, and $\sigma, \varepsilon_o \varepsilon_r, \mu_o$ the electromagnetic constitutive parameters for the fluid at rest. The induced electromagnetic fields are denoted by $\underline{E}, \underline{B}$.

By assumption, $\underline{V}$ is not an explicit function of time (steady flow). Therefore, in the "laboratory" coordinate system, with respect to which the fluid is moving, the induced static fields $\underline{E}, \underline{B}$ must satisfy the following relations [10]:

$$\nabla \times \underline{E} = 0 , \tag{1}$$

$\nabla \times (\underline{B} + \underline{B}_o) \equiv$

$$\nabla \times \underline{B} = \begin{cases} 0 \quad ; y > 0 & \text{(2a)} \\ \mu_o \sigma [\underline{E} + \underline{V} \times (\underline{B}_o + \underline{B}] + \mu_o \rho \underline{V} + \mu_o \nabla \times (\underline{P} \times \underline{V}) \; ; y < 0 , & \end{cases}$$

where (2b)

$$\underline{P} = \varepsilon_o (\varepsilon_r - 1) \, [\underline{E} + \underline{V} \times (\underline{B}_o + \underline{B})] \, . \tag{2c}$$

The quantities on the right of (2) have the following physical interpretation:

(i) $\quad \sigma[\underline{E} + \underline{V} \times (\underline{B}_o + \underline{B})]$

is the total conduction current in the fluid, which is simply $\sigma \underline{E}'$ with

$$\underline{E}' = \underline{E} + \underline{V} \times (\underline{B}_o + \underline{B}) ,$$

the electric field relative to a coordinate system that is momentarily at rest relative to the fluid.

(ii) $\rho \underline{V}$ is the convection current that arises from the spatial transfer of free electric charge by the fluid motion.

(iii) $\nabla \times (\underline{P} \times \underline{V})$ is the dielectric polarization induced current whose source is the electric field induced polarization charge transported by the fluid. This current is sometimes referred to as the Röntgen or Eichenwald current.

For $y < 0$, the constitutive relation between $\underline{E}$ and $\underline{D}$ is

$$\underline{D} = \varepsilon_o \underline{E} + \underline{P}$$

$$= \varepsilon_o \varepsilon_r \underline{E} + \varepsilon_o(\varepsilon_r - 1) \underline{V} \times (\underline{B}_o + \underline{B}) . \qquad (3)$$

The "free" charge density ρ is given by $\rho = \nabla \cdot \underline{D}$,

$$\rho = \varepsilon_o \varepsilon_r \nabla \cdot \underline{E} + \varepsilon_o(\varepsilon_r - 1) \nabla \cdot [\underline{V} \times (\underline{B}_o + \underline{B})] \qquad (4)$$

In general, in order to solve for $\underline{E}$ and $\underline{B}$, Eqs. (1-4) must be supplemented by the equations of fluid dynamics together with a specification of boundary conditions for the particular geometry. However, if the fluid velocity $\underline{V}$ of interest is sufficiently low so that the induced magnetic field is much smaller than the applied field, i.e., $|\underline{B}| << |\underline{B}_o|$, the magnetic fields $\underline{B}_o + \underline{B}$ entering on the right of (2), (3), and (4) may be replaced by the prescribed field $\underline{B}_o$, in which case the fluid velocity $\underline{V}$ and the magnetic field $\underline{B}_o$ enter only as prescribed forcing functions. This effectively decouples the fluid mechanics problem from the electromagnetics problem, i.e., the two problems can be handled independently. We then assume that the hydrodynamic problem has been solved, yielding fluid velocity fields $\underline{V}(\underline{r})$ for $y < 0$. The electromagnetic field equations now simplify to

$$\nabla \times \underline{E} = 0 \tag{5}$$

$$\nabla \times \underline{B} = \begin{cases} 0 \; ; \; y > 0 & \text{(6b)} \\ \mu_o\sigma(\underline{E} + \underline{V} \times \underline{B}_o) + \mu_o\rho\underline{V} + \mu_o\nabla \times (\underline{P} \times \underline{V}) \; ; \; y < 0 \, , & \text{(6b)} \end{cases}$$

$$\underline{P} = \varepsilon_o(\varepsilon_r - 1)\;(\underline{E} + \underline{V} \times \underline{B}_o) \; ; \; y < 0 \, , \tag{6c}$$

$$\underline{D} = \varepsilon_o\varepsilon_r\,\underline{E} + \varepsilon_o(\varepsilon_r - 1)\;\underline{V} \times \underline{B}_o \; ; \; y < 0 \, , \tag{6d}$$

$$\rho = \varepsilon_o\varepsilon_r\,\nabla \cdot \underline{E} + \varepsilon_o(\varepsilon_r - 1)\;\nabla \cdot (\underline{V} \times \underline{B}_o) \; ; \; y < 0 \, . \tag{6e}$$

Only $\underline{V}$ is prescribed for $y \leq 0$. The charge density ρ (if any) must therefore be uniquely determined from $\underline{V}$. We will now obtain a connecting relation between $\underline{V}$ and ρ. Upon taking the divergence of both sides of Eq. (6b), we obtain

$$\begin{aligned} 0 &= \sigma[\nabla \cdot \underline{E} + \nabla \cdot (\underline{V} \times \underline{B}_o)] + \nabla \cdot \rho\underline{V} \\ &= \sigma\,\nabla \cdot \underline{E} + \sigma(\nabla \times \underline{V}) \cdot \underline{B}_o + \rho\,\nabla \cdot \underline{V} + \underline{V} \cdot \nabla\rho \, . \end{aligned}$$

We denote by $\underline{\omega}$ the fluid vorticity,

$$\nabla \times \underline{V} = \underline{\omega} \tag{7a}$$

and let

$$\xi = \underline{\omega} \cdot \underline{B}_o \, . \tag{7b}$$

Also, the fluid will be treated as incompressible ($\nabla \cdot \underline{V} = 0$). One then finds

$$\nabla \cdot \underline{E} = -\xi - \frac{1}{\sigma}\,(\underline{V} \cdot \nabla\rho) \quad .$$

On the other hand, from Eq. (6e), one has

$$\rho = \varepsilon_o \varepsilon_r \nabla \cdot \underline{E} + \varepsilon_o(\varepsilon_r - 1)\xi \quad . \tag{8}$$

After eliminating $\nabla \cdot \underline{E}$ from the two preceding relations, one obtains

$$\frac{\varepsilon_o \varepsilon_r}{\sigma} (\underline{V} \cdot \nabla \rho) + \rho = -\varepsilon_o \xi \; ; \; y < 0 \quad , \tag{9}$$

which is a first-order differential equation for ρ. Let

$$\delta = \frac{\varepsilon_o \varepsilon_r}{\sigma}$$

and (10)

$$\rho = -\varepsilon_o \, f \quad .$$

The quantity δ has the physical significance of a time constant and is referred to as the relaxation time of the medium. The differential equation for f reads

$$f + \delta(\underline{V} \cdot \nabla f) = \xi \quad ,$$

or

$$[1 + \delta(\underline{V} \cdot \nabla)] \, f = \xi \; .$$

If

$$|\delta(\underline{V} \cdot \nabla \xi)| < 1 \quad ,$$

and we also assume that ξ possesses partial derivatives of all orders with respect to x, y, z, the solution for f may be expanded in the Taylor series as follows.*

$$f = \sum_{n=0}^{\infty} (-1)^n \, \delta^n (\underline{V} \cdot \nabla)^n \, \xi \; .$$

Consequently, the free charge is given by

---

*The assumption that ξ possesses derivatives of all orders is made here for convenience only, and is not at all necessary for the validity of the final approximation as given by Eq. (15). From elementary theory of first-order partial differential equations, it can be shown that the charge density ρ is given explicitly and exactly by the following expression (continued)

$$\rho = \begin{cases} 0 \; ; \; y > 0 \, , \\ -\epsilon_o \sum_{n=0}^{\infty} (-1)^n \delta^n (\underline{V} \cdot \nabla)^n \xi \; ; \; y < 0 \, . \end{cases} \tag{11}$$

Returning now to (8) and solving for $\nabla \cdot \underline{E}$, we obtain

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_o \epsilon_r} - \left(\frac{\epsilon_r - 1}{\epsilon_r} \right) \xi \; ; \; y < 0 \, ,$$

Substituting for ρ from (11) yields

$$\nabla \cdot \underline{E} = -\frac{1}{\epsilon_r} \sum_{n=0}^{\infty} (-1)^n \delta^n (\underline{V} \cdot \nabla)^n \xi - \left(\frac{\epsilon_r - 1}{\epsilon_r} \right) \xi \, ,$$

which is equivalent to

$$\nabla \cdot \underline{E} = -\xi - \frac{1}{\epsilon_r} \sum_{n=1}^{\infty} (-1)^n \delta^n (\underline{V} \cdot \nabla)^n \xi \, . \tag{12}$$

By virtue of (5) one may set

$$\underline{E} = -\nabla \phi \, ,$$

and (12) becomes the Poisson equation for the scalar potential:

$$\nabla^2 \phi = \begin{cases} 0 \; ; \; y > 0 \\ \xi + \frac{1}{\epsilon_r} \sum_{n=1}^{\infty} (-1)^n \delta^n (\underline{V} \cdot \nabla)^n \xi \; ; \; y < 0 \, . \end{cases} \tag{13}$$

(continued)

$$\rho = \rho(0) e^{-t/\delta} - \epsilon_o \xi(t) + \delta \, \epsilon_o e^{-t/\delta} \int_0^{t/\delta} \xi'(s\delta) \, e^s \, ds \, ,$$

where t is a parameter measured along the fluid particle trajectory and $\xi' \equiv d\xi/dt$. From this result follows immediately that for any $t > 0$, $\lim \rho \to -\epsilon_o \xi$ as $\delta \to 0$, provided only that $|\xi'|$ is uniformly bounded.

Thus, given a fluid velocity field $\underline{V}$, (13) may be solved for ϕ, subject to the appropriate set of boundary conditions. Once the electrostatic field has been determined, the driving function (i.e., the equivalent current) in the generalized Ampere's law statement (6b) can be expressed completely in terms of known quantities. Thus we have

$$\nabla \times \underline{B} = \mu_o \underline{J}_e , \tag{14a}$$

$$\underline{J}_e = \sigma(\underline{E} + \underline{V} \times \underline{B}_o) - \varepsilon_o \underline{V} \sum_{n=0}^{\infty} (-1)^n \delta^n (\underline{V} \cdot \nabla)^n \xi$$

$$+ \varepsilon_o(\varepsilon_r - 1) \nabla \times [\underline{E} \times \underline{V} - \underline{V}(\underline{V} \cdot \underline{B}_o) + \underline{B}_o V^2] \quad . \tag{14b}$$

Since ξ depends on $\underline{V}$, the source terms giving rise to the scalar potential in (13) as well as the equivalent current in Ampere's law (14) are nonlinear functions of $\underline{V}$. Although such nonlinear dependence on $\underline{V}$ may well be of great interest for poorly conducting media, they are of no consequence for sea water where δ is very small. Thus, since $\varepsilon_o = 1/36\pi \times 10^{-9}$, $\delta = \varepsilon_o \varepsilon_r / \sigma$ is small for all but good insulators. In particular, for sea water, $\sigma \doteq 4$, $\varepsilon_r \approx 80$ so that $\varepsilon_o \varepsilon_r / \sigma$ is certainly a small quantity. Thus, in the series expansion for ρ in (11) we need not bother with terms for $n > 0$, and write, to a good approximation,

$$\rho = \begin{cases} 0 \; ; \; y > 0 & (15a) \\ -\varepsilon_o \xi \; ; \; y < 0 \; . & (15b) \end{cases}$$

For the magnetic field, this approximation amounts to dropping all terms of order ε_o on the right of Eq. (14b). We then have $\underline{B} = \mu_o \underline{H}$, and

$$\nabla \times \underline{H} = \begin{cases} 0 \; ; \; y > 0 \quad , & (16a) \\ \sigma(-\nabla\phi + \underline{V} \times \underline{B}_o) \; , \; y < 0 \quad . & (16b) \end{cases}$$

The scalar potential is given by

$$\nabla^2\phi = \begin{cases} 0 \; ; \; y > 0 \\ \underline{\omega} \cdot \underline{B}_o = \xi \; ; \; y < 0 \; . \end{cases} \tag{17}$$

B. THE ELECTROSTATIC FIELD

We shall first obtain formulas for the electrostatic field. It must be borne in mind, however, that all subsequent results based on (17) need not apply for arbitrary σ, in particular for $\sigma = 0$. Equation (17) is indeed somewhat peculiar in that nowhere does there appear a dependence on the medium parameters. Our guide as to boundary conditions must be Maxwell equations (5) and (16). The first boundary condition is dictated by (5), which requires that ϕ be continuous at $y = 0$, i.e.,

$$\phi \Big|_{y = 0^-} = \phi \Big|_{y = 0^+} \; . \tag{18}$$

The second boundary condition follows from (16), which demands continuity of the normal component of the total conduction current $\sigma(\underline{E} + \underline{V} \times \underline{B}_o)$ at $y = 0$. Since the normal component of this current is identically zero at $y = 0^+$, we must also have

$$-\frac{\partial\phi}{\partial y} + V_z B_{ox} - V_x B_{oz} = 0 \quad \text{at } y = 0^- \; . \tag{19}$$

In solving (17) subject to (18) and (19), it is instructive to consider separately rotational and irrotational flow. For the latter $\underline{\omega} \equiv 0$, so that for $y \leq 0$, (17) reduces to the Laplace equation with a prescribed normal derivative of the potential at $y = 0^-$. The solution for ϕ for $y < 0$ is easily shown to be

$$\phi(x,y,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; \frac{V_z(x',0,z')\, B_{ox} - V_x(x',0,z')\, B_{oz}}{\sqrt{(x-x')^2 + (z-z')^2 + y^2}} . \tag{20}$$

Equation (20) also satisfies the Laplace equation for $y > 0$. Moreover, ϕ as given by (20) is continuous at $y = 0$. [Boundary condition (18).] Consequently, (20) is the complete solution valid for $-\infty < y < \infty$. It may be shown directly from (20) that

$$\lim_{y \to 0^-} \frac{\partial \phi}{\partial y} \to V_z(x,0,z)\, B_{ox} - V_x(x,0,z)\, B_{oz} \tag{21}$$

which is just the prescribed boundary condition (19). On the other hand, when the limit is approached from the positive y direction, the result is

$$\lim_{y \to 0^+} \frac{\partial \phi}{\partial y} \to - V_z(x,0,z)\, B_{ox} + V_x(x,0,z)\, B_{oz} , \tag{22}$$

which is the negative of Eq. (21). Hence $\frac{\partial \phi}{\partial y}$ is discontinuous across $y = 0$ by twice the value prescribed by* Eq. (19).

*This can also be deduced by a symmetry argument: ϕ is an even function of y, therefore $\frac{\partial \phi}{\partial y}$ must be odd. Since $\frac{\partial \phi}{\partial y}$ is also discontinuous, half of the jump must occur for $y = 0^-$ and half for $y = 0^+$

With the aid of Eqs. (21) and (22) we can compute the total surface charge at the interface:

$$\rho_s = -\varepsilon_o \frac{\partial\phi}{\partial y}\bigg|_{y=0^+} - D_y\bigg|_{y=0^-} =$$

$$= 2\varepsilon_o[V_z(x,0,z)\ B_{ox} - V_x(x,0,z)\ B_{oz}]\ . \qquad (23)$$

Note that ε_r does not enter into the expression for charge. This is a consequence of the approximation for the highly conducting medium (Eq. 15).

Thus, for purely irrotational flows, the electrostatic field above and below the ocean surface depends only on the tangential components of velocity at the surface. In this sense, it can be considered a pure surface phenomenon. One other point is worth mentioning: formula (20) for the potential does not depend on conductivity or any other parameter of the medium. From this, one should not conclude that an electric field will be induced by a flow in a medium with zero conductivity. For as has been pointed out in the discussion preceding (17), in the limit of low conductivity (17) no longer applies since in that case nonlinear effects associated with the convection current begin to dominate.

Next we consider the case of rotational flow. Unlike in the case of pure potential flow, the right side of (17) will not be zero. The formal solution for ϕ will now contain a volume integral of the product of ξ and a suitable Green's function. We shall denote this contribution to the potential by $\hat{\phi}$. From the linearity of the problem, it follows that we can superpose the solution (20) and the solution to the inhomogeneous problem with the boundary condtion (18) together with the additional condition

$$\frac{\partial\hat{\phi}}{\partial y} = 0 \quad \text{at } y = 0^- . \tag{24}$$

For $y < 0$ we then have an inhomogeneous Neumann problem, so that

$$\hat{\phi}(x,y,z) = -\int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; G_N(x,y,z;x',y',z') \; \xi(x',y',z') \; ; \; y < 0 \tag{25}$$

where G_N is the Neumann Green's function given by

$$G_N(x,y,z;x',y',z') = G_o(x,y,z;x',y',z') + G_o(x,y,z;x',-y',z') \tag{26}$$

where

$$G_o(x,y,z;x',y',z') = \frac{1}{4\pi}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-\frac{1}{2}} . \tag{27}$$

(Note that in (26) y' and y are less than zero.) To obtain $\hat{\phi}$ in the region $y > 0$, we utilize the fact that ϕ must be continuous at $y = 0$. Clearly, for $y > 0$

$$\hat{\phi}(x,y,z) = -2\int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; G_o(x,y,z;x',y',z') \; \xi(x',y',z') \; , \tag{28}$$

for it reduces to (25) at $y = 0$ and satisfies the Laplace equation for $y > 0$.

The complete solution for the scalar potential is then given by the sum of (20) and (28) or (25). Thus, for $y > 0$,

$$\phi(x,y,z) = 2\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; G_o(x,y,z;x',0,z') \, [V_z(x',0,z') \, B_{ox} - V_x(x',0,z') \, B_{oz}]$$

$$-2\int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; G_o(x,y,z;x',y',z') \, \underline{\omega}(x',y',z') \cdot \underline{B}_o \, , \tag{29}$$

while for $y < 0$

$$\phi(x,y,z) = 2\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; G_o(x,y,z;x',0,z') \, [V_z(x',0,z') \, B_{ox} - V_x(x',0,z') \, B_{oz}]$$

$$-\int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; [G_o(x,y,z;x',y',z') + G_o(x,y,z;x',-y',z')] \cdot [\underline{\omega}(x',y',z') \cdot \underline{B}_o] \, . \tag{30}$$

Equations (29) and (30) are valid generally for steady flows, i.e., the flow can be partly rotational and partly irrotational. For purely irrotational flow the volume contributions vanish, and one again obtains (20). It is perhaps worth remarking that at this point no explicit assumptions have been made with regard to boundary conditions to be satisfied by the fluid velocity fields. Of course, the assumption of a perfectly planar surface implies that the normal component of the fluid velocity must vanish immediately

below the surface.* However, thus far we made use of the planar nature of the surface only in the electrostatic part of the problem.

Next we consider the special case of pure rotational flow in which the normal component of fluid velocity vanishes at the ocean surface.** Quite generally, as long as we are dealing with incompressible fluids, we can express the velocity generated by a distribution of vorticity as the curl of a hydrodynamic vector potential $\underline{\psi}$ (vector stream function).

Thus, with

$$\underline{V} = \nabla \times \underline{\psi} \quad , \tag{31}$$

the vorticity $\underline{\omega}$ is given by

$$\underline{\omega} = \nabla \times \nabla \times \underline{\psi} \quad . \tag{32}$$

Since the last is equivalent to

$$\nabla^2 \underline{\psi} - \nabla\nabla \cdot \underline{\psi} = -\underline{\omega} \quad , \tag{33}$$

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*This is, of course, not strictly compatible with surface wave phenomena in the ocean where a vertical velocity of the surface is necessary to sustain any kind of surface wave activity. For small amplitude (linear) surface waves, this vertical velocity is treated as a small perturbation of an otherwise planar surface in which case this surface may also be treated as planar in the electromagnetic problem.

**This boundary condition is usually adopted in modeling internal wave phenomena.

we have a differential equation for $\underline{\psi}$ with the vorticity function playing the role of a source. Since

$$\nabla \cdot \underline{\omega} = 0 \tag{34}$$

we can always choose the gauge

$$\nabla \cdot \underline{\psi} = 0 \tag{35}$$

which gives for (33)

$$\nabla^2 \underline{\psi} = -\underline{\omega} \ . \tag{36}$$

It is important to keep in mind that "arbitrary" vector vorticity sources may not be prescribed on the right of (36) but only those having zero divergence (34). Otherwise the solution of (36) for $\underline{\psi}$ will not satisfy (35). We now assume that the vertical motion of the interface can be neglected so that the boundary condition on $\underline{V}$ is

$$V_y = 0 \text{ at } y = 0. \tag{37}$$

Using (37) in conjunction with (31) and (35) leads to two boundary conditions on $\underline{\psi}$ at $y = 0$. Thus, Eq. (31) demands that the vertical curl of the velocity be zero, which can be satisfied only if

$$\psi_x = \psi_z = \text{constant at } y = 0 \ . \tag{38a}$$

Since the value of the velocity field as computed from (31) is unaffected by the addition of a constant to the stream function, we may set this constant to zero. The gauge condition, Eq. (35) then yields

$$\frac{\partial \psi_y}{\partial y} = 0 \text{ at } y = 0 \ . \tag{38b}$$

With the aid of (36) and (38) the general equations (29) and (30) can be put into a form which involves only volume integrals. Instead of doing this directly, we will follow a procecure which closely parallels that found in the published

literature [7]. This involves combining the hydrodynamic equation (36) with the equation for the electrostatic potential Eq. (17):

$$\nabla^2(\underline{B}_o \cdot \underline{\psi} + \phi) = 0 \; ; \; y < 0 \cdot \tag{39}$$

If we now set

$$\phi' = \underline{B}_o \cdot \underline{\psi} + \phi \quad , \tag{40}$$

ϕ' satisfies the Laplace equation

$$\nabla^2\phi' = 0 \; ; \; -\infty < y < \infty \quad , \tag{41}$$

and the solution for ϕ is

$$\phi = \begin{cases} \phi' & ; \quad y > 0 \quad , & (42a) \\ \phi' - \underline{B}_o \cdot \underline{\psi} & ; \; y < 0 \quad . & (42b) \end{cases}$$

This form is employed in [7] where ϕ' is set equal to zero so that the electrostatic field above the ocean surface is identically zero. As will be shown in the sequel $\phi' = 0$ is compatible only with a rotational flow with zero vertical component of vorticity. We shall elaborate on this point after we have set up a general solution for ϕ'. The solution of (41) is completely determined by the boundary conditions at $y = 0$. From (19) and (40) one finds for $y = 0^-$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi'}{\partial y} - \frac{\partial}{\partial y} \underline{B}_o \cdot \underline{\psi} = V_z B_{ox} - V_x B_{oz} \quad . \tag{43}$$

From the definition of the stream function (31)

$$V_x = \frac{\partial\psi_z}{\partial y} - \frac{\partial\psi_y}{\partial z} \qquad V_z = \frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y} \quad .$$

Substituting for V_x and V_z in (43) and taking account of (38b) one obtains

$$\frac{\partial\phi'}{\partial y} = B_{ox} \frac{\partial\psi_y}{\partial x} + B_{oz} \frac{\partial\psi_y}{\partial z} \quad \text{at} \quad y = 0^- \, . \tag{44}$$

The second boundary condition on ϕ' follows from (18) and (42):

$$\phi'\Big|_{y = 0^+} - \phi'\Big|_{y = 0^-} = -B_{oy} \psi_y \tag{45}$$

where we have taken account of (38a).

Although (44) and (45) suffice to write down the complete solution for ϕ', we prefer to decompose ϕ' into a sum of three parts, each arising from one of the three components of $\underline{B}_o$, and then add the result. In this manner the simplifications in the final formulae that arise from a particular orientation of $\underline{B}_o$ and the vorticity function are best brought in evidence.

Accordingly, we denote by ϕ'_x, ϕ'_y, ϕ'_z the potential functions due to the x, y, z components of $\underline{B}_o$,

respectively, and write

$$\phi' = \phi'_x + \phi'_y + \phi'_z \quad . \tag{46}$$

First, let $B_{ox} = B_{oy} = 0$. Then from (45) ϕ'_z is continuous at $y = 0$ while its normal derivative at $y = 0^-$ is $B_{oz} \frac{\partial \psi_y}{\partial z}$. The potential ϕ'_z is then given by a formula similar to (20), viz.,

$$\phi'_z(x,y,z) = \frac{B_{oz}}{2\pi} \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dz'' \; \frac{\partial \psi_y(x'',o,z'')}{\partial z''} [(x-x'')^2 + (z-z'')^2 + y^2]^{-\frac{1}{2}} \quad , \tag{47}$$

which holds for $-\infty < y < \infty$ by virtue of the continuity of ϕ' at $y = 0$. We would like to express the final result in terms of the volume distribution of the y component of vorticity. This can be done by first solving for ψ_y in (36) and substituting for $\frac{\partial \psi_y}{\partial z''}$ in the integrand of (47). By virtue of the boundary condition (38b) the solution for ψ_y in terms of ω_y reads

$$\psi_y(x'',y'',z'') = \int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; G_N(x'',y'',z'';x',y',x') \, \omega_y(x',y',z') \tag{48}$$

where G_N is the Neumann-type Green's function defined in (26) and $y'' \leq 0$. After (48) is differentiated with respect to z'' and substituted in (47) the integration with respect to x'' and z'' can be carried out, leaving a three-fold integral over the source coordinates x',y',z'. While the computation is straightforward, it is somewhat lengthy and has therefore been

relegated to Appendix B. From Eqs. (B-1) and (B-11) the result is

$$\phi_z'(x,y,z) = B_{oz}\int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; K_z(x,y,z;x',y',z')\,\omega_y(x',y',z') \tag{49}$$

where

$$K_z(x,y,z,x',y',z') = -\frac{1}{2\pi|\underline{\rho}-\underline{\rho}'|}\left(\frac{(z-z')}{|\underline{\rho}-\underline{\rho}'|}\right)\left(1 - \frac{|y\mp y'|}{\sqrt{|\underline{\rho}-\underline{\rho}'|^2+(y\mp y)^2}}\right) \tag{50}$$

and

$$|\underline{\rho}-\underline{\rho}'| = \sqrt{(x-x')^2 + (z-z')^2}$$

and where the (−) and (+) signs refer to $y > 0$ and $y < 0$, respectively.

Next we set $B_{oy} = B_{oz} = 0$ and compute ϕ_x'. From (45) we again find that ϕ_x' is continuous while at $y = 0^-$ $\frac{\partial\phi_x'}{\partial y} = B_{ox}\frac{\partial\psi_y}{\partial x}$. Hence

$$\phi_x'(x,y,z) = \frac{B_{ox}}{2\pi}\int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dz'' \; \frac{\partial\psi_y(x'',0,z'')}{\partial x''}\,[(x-x'')^2 + (z-z'')^2 + y^2]^{-\frac{1}{2}}\,. \tag{51}$$

Again ψ_y is given by (48), and the final expression for ϕ_x' in terms of the vorticity function may be written as in (49):

$$\phi_x'(y,z,z) = B_{ox}\int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; K_x(x,y,z;x',y',z') \; \omega_y(x',y',z'). \tag{52}$$

The expression for K_x may be written down from (50) by simply interchanging $(z-z')$ with $(x-x')$. Thus, one obtains

$$K_x(x,y,z;x',y',z') = -\frac{1}{2\pi|\underline{\rho}-\underline{\rho}'|}\left(\frac{x-x'}{|\underline{\rho}-\underline{\rho}'|}\right)\left(1 - \frac{|y\mp y'|}{\sqrt{|\underline{\rho}-\underline{\rho}'|^2 + (y\mp y')^2}}\right). \tag{53}$$

For the third and final case, viz., $B_{ox} = B_{oz} = 0$, we have the following boundary conditions on ϕ_y'. From (44)

$$\frac{\partial\phi_y'}{\partial y} = 0 \quad \text{at} \quad y = 0^- \; , \tag{54}$$

while from (45) ϕ_y' is discontinuous across $y = 0$ by the amount*

$$\phi_y'\Big|_{y=0^+} - \phi_y'\Big|_{y=0^-} = -B_{oy}\,\psi_y(x,0,z)\,. \tag{55}$$

*This discontinuity produces no anomalies (infinite voltage, etc.,) since by virtue of (42) the true electrostatic potential is continuous at $y = 0$.

This case is therefore distinctly different from the two previous cases. First, since we are dealing with the homogeneous Laplace equation, (54) demands that for $y < 0$ $\phi'_y = 0$ (or a constant, which we are at liberty to set equal to zero, since we are not interested in the absolute value of the potential). For $y > 0$, we obtain ϕ'_y, subject to the boundary condition $\phi'_y\big|_{y=0^+} = B_{oy}\ \psi_y(x,0,z)$. The solution to this standard problem is

$$\phi'_y(x,y,z) = 2B_{oy}\int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dz''\ \frac{\partial}{\partial y}\ G_o(x,y,z;x'',0,z'')\ \psi_y(x'',0,z''). \tag{56}$$

Again, we employ (48) and write the final result

$$\phi'_y(x,y,z) = B_{oy}\int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz'\ K_y(x,y,z;x',y',z')\ \omega_y(x',y',z') \tag{57}$$

where, as shown in Appendix B, Eq. (B-17),

$$K_y(x,y,z;x',y',z') = \begin{cases} -\frac{1}{2\pi}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-\frac{1}{2}} ; & y > 0 \\ 0 ; & y < 0 \end{cases} \tag{58}$$

We have defined $K_y = 0$ for $y < 0$ so that formula (57) automatically encompasses $y < 0$ where ϕ' vanishes. If we define a vector $\underline{K}(x,y,z;x',y',z')$ with components given by (50), (53) and (58), we can write the true electrostatic potential in (42) as follows

$$\phi(x,y,z) = \int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; \underline{B}_o \cdot \underline{K}(x,y,z;x',y',z') \, \omega_y(x',y',z') \; ; \; y > 0 \, , \tag{59a}$$

$$\phi(x,y,z) = \int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' \; \underline{B}_o \cdot \underline{K}(x,y,z;x',y',z') \, \omega_y(x',y',z') - \underline{B}_o \cdot \underline{\psi}(x,y,z) \; ; \; y < 0 \, . \tag{59b}$$

Equation (59) gives the electrostatic potential above and below the ocean surface when the flow is rotational and for which the normal component of fluid velocity vanishes at the ocean surface. We observe that unlike in the case of potential flow, Eq. (20), the generation of electric fields by pure rotational flow is a "volume phenomenon", i.e., ϕ depends on the distribution of the vector vorticity function everywhere below the surface. Indeed, (59) is nothing but the transformed general Eqs. (29)(30) specialized to pure rotational flow. Evidently for pure rotational flow, the surface terms can be expressed as integrals over the vorticity function. Indeed, all that was done in arriving at (59) was to cast these "volume" contributions into a special form. There are certain features that are obscured by (30), but are brought out explicitly by (59). For one, we notice that the electrostatic potential above the ocean surface arises entirely from the vertical component of vorticity. Thus, unless there is a nonzero vertical vorticity component, that portion of the electrostatic field that is induced by rotational flow vanishes identically above the ocean surface. In this case the electrostatic potential below the surface is just the negative of the scalar product of the earth's magnetic field and the vector stream function.

Consider now rotational flow in which the vorticity function is purely horizontal (in which case Eq. (59) yields no electrostatic field above the ocean surface). We first show that in this case it is not possible to construct a vorticity function that is unidirectional unless it exhibits no variation along this direction. For without loss of generality we may assume this direction to be the z-direction and we have $\underline{\omega} = \underline{z}_0 \omega_z$. Since, by definition, $\nabla \cdot \underline{\omega} = 0$ [this condition was used in arriving at Eq. (59)], $\frac{\partial \omega_z}{\partial z} = 0$, which clearly shows that ω_z cannot vary with z. Consequently, a single *unidirectional* vorticity component is consistent only with a two-dimensional problem.

We now consider an important special quasi-two-dimensional problem. We suppose a purely rotational flow problem in which there are only two nonzero velocity components $V_x(x,y,z)$, $V_y(x,y,z)$. The components of vorticity are

$$\omega_z = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \quad , \tag{60a}$$

$$\omega_y = \frac{\partial V_x}{\partial z} \quad , \tag{60b}$$

$$\omega_x = - \frac{\partial V_y}{\partial z} \quad .$$

If the variation of the velocity fields with the longitudinal (z) direction is small, then ω_y and ω_x will be small, and the flow field would be expected to resemble that obtained for a strictly two-dimensional flow pattern in which $\omega_y = \omega_x = 0$. Although such an approximation may be adequate to describe the major features of the hydrodynamic problem, the same cannot be

said with regard to the computation of the electrostatic potential. For the potential depends on the integral involving $\frac{\partial V_x}{\partial z}$, taken over the whole fluid volume. In particular, the integration in Eq. (59) extends over the entire length (z-direction) of the flow field, so that locally small longitudinal gradients of the horizontal velocity do not necessarily imply that their integrated effects will also be small.

If a purely horizontal vorticity function is not unidirectional, then the two horizontal components ω_x, ω_z must satisfy

$$\frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_z}{\partial z} = 0 \quad ,$$

i.e., the transverse divergence of $\underline{\omega}$ vanishes. This condition is actually satisfied by linear internal waves at frequencies substantially above the inertial frequency (i.e., in the absence of Coriolis effects). Internal waves under these circumstances will not induce any electric field above the ocean surface. By contrast, surface waves necessarily give rise to electric fields above the ocean surface. The corresponding electrostatic* potential being given by (20).

C. THE MAGNETOSTATIC FIELD

Having determined the electrostatic fields, the magnetostatic fields are determined from (16) subject to the continuity of tangential components of $\underline{H}$ across the interface $y = 0$. Also, since the magnetic properties of air and sea water are essentially identical, the normal component of $\underline{H}$ at $y = 0$ must be continuous as well. We solve (16) for $\underline{H}$ by introducing the Lorentz vector potential $\underline{A}$,

*Here we are jumping ahead of the story since we have thus far considered only the purely static case.

$$\mu_o \underline{H} = \nabla \times \underline{A} \tag{61}$$

and choose the Coulomb gauge

$$\nabla \cdot \underline{A} = 0 . \tag{62}$$

Substituting for $\underline{H}$ in terms of $\underline{A}$ in Eq. (16) and employing the identity $\nabla\times\nabla\times = \nabla\nabla\cdot - \nabla^2$ we obtain

$$\nabla^2 \underline{A} = \begin{cases} 0 \ ; \ y > 0 , \\ -\sigma\mu_o(-\nabla\phi + \underline{V} \times \underline{B}_o) \ ; \ y < 0 \ . \end{cases} \tag{63}$$

From (16), (61), and (62) follows that the boundary conditions on $\underline{A}$ are continuity of each component and its normal derivative. This would also be the case, for example, if the equivalent current density $\underline{J}_e$,

$$\underline{J}_e = \sigma(-\nabla\phi + \underline{V} \times \underline{B}_o) , \tag{64}$$

were prescribed in free space. Since, moreover, we constructed our scalar potential such as to ensure that $\nabla\cdot\underline{J}_e = 0$ everywhere, (including at the boundary) the problem posed in (63) can indeed be solved with the aid of the free-space Green's function. Hence,

$$\underline{A}(x,y,z) = \mu_o \int_{-\infty}^{0} dy' \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dx' \ \underline{J}_e(x',y',z') \ G_o(x,y,z;x',y',z') . \tag{65}$$

The vector potential $\underline{A}$ arises from two partial contributions: the direct source contribution from the Lorentz force term

$$\underline{J}^{(s)} = \sigma(\underline{V} \times \underline{B}_o) \quad , \tag{66}$$

and the contribution from the conduction current whose source is the static electric field $-\nabla\phi$ below the ocean surface. Evidently, since ϕ as given by Eq. (30) involves a volume integral, the partial contribution to $\underline{A}$ from $-\sigma\nabla\phi$ in (65) requires two-volume integrals over the fluid velocity components. This two-fold integration can be reduced to a single integral and the total vector potential can then be expressed as a single integral involving only $\underline{J}^{(s)}$ in (66). That such a representation should be possible is evident from the fact that $\sigma(\underline{V} \times \underline{B}_o)$ is the primary source (excitation) of the electromagnetic fields. However, the Green's function kernel entering into such a representation of $\underline{A}$ will no longer be G_o as in (65).

Instead of carrying out the rather cumbersome steps of reducing the double-volume integral involving the electrostatic potential contribution to a single-volume integral, we shall obtain the final result by an alternate route.

Clearly, the distinction between the representation (65) and any alternate one is in the choice of the gauge condition. For example, (65) is a consequence of adhering to the Coulomb gauge, Eq. (62). Alternatively, we could have employed the Lorentz gauge:

$$\nabla \cdot \underline{A} = \begin{cases} 0 \; ; \; y > 0 \; , \\ -\mu_o \sigma\phi \; ; \; y < 0 \; . \end{cases} \tag{67}$$

The detailed derivation of the fields for this choice of gauge is presented in Appendix C.

As a consequence of this choice, the vector potential $\underline{A}$ now satisfies

$$\nabla^2 \underline{A} = \begin{cases} 0 \; ; \; y > 0 \\ \\ -\mu_o \underline{J}^{(s)} \; ; \; y < 0 \; . \end{cases} \tag{68}$$

The source of the vector potential now comprises only the direct Lorentz forcing term $\underline{J}^{(s)} = \sigma(\underline{V} \times B_o)$, and the solution for $\underline{A}$ will now involve only a single integral over $\mu_o \underline{J}^{(s)}$ weighted with the appropriate Green's function. However, this Green's function is no longer G_o as in (65), but a more complicated Tensor quantity. This arises from the fact that the boundary conditions on $\underline{A}$ in (68) at the planar interface are no longer the same as those for a vector potential generated by a prescribed current distribution in free space. While the tangential components (A_x, A_z) and their normal derivatives are still continuous at $y = 0$ (just as in (65)), the normal derivative of A_y is discontinuous at the interface. The specific expression for $\underline{A}$ is

$$\underline{A}(\underline{r}) = \mu_o \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2\underline{\rho}' \;\; \underline{\underline{G}}(\underline{r},\underline{r}') \; \underline{J}^{(s)}(\underline{r}') \tag{69}$$

where $\underline{\underline{G}}(\underline{r},\underline{r}')$ is the Tensor Green's function with the matrix representation

$$\underline{\underline{G}}(\underline{r},\underline{r}') = \begin{bmatrix} G_o & 0 & 0 \\ G_{yx} & G_{yy} & G_{yz} \\ 0 & 0 & G_o \end{bmatrix} , \qquad (70)$$

where G_o is again the free space Green's function. The other non-zero components of $\underline{\underline{G}}$ are

$$G_{yx}(\underline{r},\underline{r}') = -\frac{1}{4\pi} \frac{(x-x')\left[\sqrt{(\underline{\rho}-\underline{\rho}')^2 + (y\mp y')^2} - |y\mp y'|\right]}{|\underline{\rho}-\underline{\rho}'|^2 \sqrt{|\underline{\rho}-\underline{\rho}'|^2 + (y\mp y')^2}} , \quad (71a)$$

$$G_{yz}(\underline{r},\underline{r}') = -\frac{1}{4\pi} \frac{(z-z')\left[\sqrt{(\underline{\rho}-\underline{\rho}')^2 + (y\mp y')^2} - |y\mp y'|\right]}{|\underline{\rho}-\underline{\rho}'|^2 \sqrt{|\underline{\rho}-\underline{\rho}'|^2 + (y\mp y')^2}} , \quad (71b)$$

$$G_{yy}(\underline{r},\underline{r}') = \begin{cases} 0 \; ; \; y > 0 \\ G_o(x,y,z;x',y',z') - G_o(x,y,z;x',-y',z') ; \; y < 0 \end{cases} , \quad (71c)$$

where the minus and plus sign in (71a,b) pertains to observation points $y > 0$ and $y < 0$, respectively.

From (70) it is evident that the two horizontal components of $\underline{A}$ are the same as those that would be given by (65) were the

electric field contribution to $\underline{J}_e$ to be omitted. Consequently, A_x and A_z (and hence the vertical component of the induced magnetic field) are not affected by the electric current component generated directly by the electric field; the only component of $\underline{A}$ that depends on the subsurface electric field is A_y (which contributes only to the horizontal component of the induced magnetic field).

Because of the use of the Lorentz gauge, the electric field can be obtained explicitly in terms of $\underline{A}$. Thus, the electrostatic potential is given by

$$\phi(x,y,z) = \begin{cases} -\dfrac{1}{\mu_0\sigma}\ \nabla\cdot\underline{A}\ ; & y < 0 \\[2ex] -\dfrac{2}{\mu_0\sigma}\displaystyle\iint_{-\infty}^{\infty} d^2\rho'\ \frac{\partial}{\partial y}\ G_0(x,y,z;x',0,z')\ \nabla'\cdot\underline{A}\Big]_{y'=0} = 0; & y > 0, \end{cases} \tag{72}$$

where ∇' is the gradient operator with respect to (x',y',z'). In this form the expression for the scalar potential appears substantially more complicated than the results obtained with the aid of a direct solution of the Poisson equation, viz., Eqs. (29) and (30). Thus, while use of the Lorentz gauge leads more directly to the final expressions for the magnetic field than the use of the Coulomb gauge, the relative difficulty is reversed for the electric field. It may be shown that Eq. (72) reduces, as indeed it must, to the expressions for ϕ given in (29) and (30). These equations, together with the expressions for the vector potential $\underline{A}$, Eqs. (69-71), provide a complete set of relations for determining the electrostatic and magnetostatic fields generated by steady flow.

III. ELECTROMAGNETIC FIELDS INDUCED BY TIME-DEPENDENT OCEAN CURRENTS

When the fluid velocity depends explicitly on time (non-steady flow) the electromagnetic fields are no longer static, and Eq. (5) and Eq. (16) must be replaced by

$$\nabla \times \underline{E} = -\mu_o \frac{\partial H}{\partial t} , \tag{73a}$$

$$\nabla \times \underline{H} = \begin{cases} \sigma[\underline{E} + \underline{V} \times \underline{B}_o] \; ; \; y < 0 , \\ \varepsilon_o \dfrac{\partial \underline{E}}{\partial t} \; ; \; y > 0 , \end{cases} \tag{73b}$$

where we have neglected both the displacement current and the convection current below the ocean surface. For time scales of 1 sec or longer

$$\left| \varepsilon_o \varepsilon_r \frac{\partial \underline{E}}{\partial t} \right| << \left| \sigma \underline{E} \right| , \tag{74}$$

so that the displacement current below the ocean surface can be safely neglected. As was demonstrated in the preceding section, the convection current is of the order $\underline{E} \times O(\varepsilon_o)$, and, therefore also quite negligible by comparison with $\sigma\underline{E}$. If, in addition, we are only interested in the dominant field components, then, above the ocean surface, the displacement current

term and the effects of magnetic induction can also be neglected*. On the other hand in order to understand the process of electromagnetic *power transfer* above the ocean surface, these terms must be included. An exact formulation is presented in Appendix D, and the results are employed in Chapter V-D in the discussion of electromagnetic power transport above the ocean surface.

With the displacement current and magnetic induction terms omitted for $y > 0$, we have

$$\nabla \times \underline{E} = \begin{cases} -\mu_0 \dfrac{\partial \underline{H}}{\partial t} \quad ; \quad y < 0 \,, & (75a) \\ 0 \quad ; \quad y > 0 \,. & (75b) \end{cases}$$

*Formally, this may be motivated as follows: Since $\nabla \cdot \underline{H} = 0$ everywhere, then above the ocean surface, $\underline{H}$ satisfies the wave equation

$$\nabla^2 \underline{H} = \frac{1}{c^2} \frac{\partial^2 \underline{H}}{\partial t^2} \quad ,$$

$c = (\mu_0 \varepsilon_0)^{-\frac{1}{2}}$ being the speed of light in vacuo. The solution for $\underline{H}$ can be written as a Fourier integral with respect to the transverse coordinates, viz.,

$$\underline{H}(\underline{r}) = \iint_{-\infty}^{\infty} d^2\underline{\kappa} \; e^{-i\underline{\kappa}\cdot\underline{\rho} + i\omega t} \; \underline{\mathcal{H}}(\underline{\kappa}, y) \quad ,$$

where $\underline{\mathcal{H}}$ satisfies

$$\frac{d^2 \mathcal{H}}{dy^2} + \left(\frac{\omega^2}{c^2} - \kappa^2 \right) \underline{\mathcal{H}} = 0 \quad .$$

Clearly, if $\frac{\omega}{c} << \kappa$ (spatial scales of hydrodynamic disturbances much shorter than the electromagnetic wavelength), one can set $c \to \infty$, or, which is the same thing, $\varepsilon_0 \to 0$, $\mu_0 \to 0$.

$$\nabla \times \underline{H} = \begin{cases} \sigma(\underline{E} + \underline{V} \times \underline{B}_o) & ; y < 0 , \quad (76a) \\ 0 & ; y > 0 . \quad (76b) \end{cases}$$

We now examine the quantitative significance of the magnetic induction term below the ocean surface. As usual, we express the solution of Eqs. (75a) and (76a) in terms of potentials $\underline{A}$ and ϕ:

$$\mu_o \underline{H} = \nabla \times \underline{A} , \quad (77a)$$

$$\underline{E} = -\nabla\phi - \frac{\partial A}{\partial t} . \quad (77b)$$

With aid of Eqs. (17) and (77b) we find

$$\nabla^2\phi + \frac{\partial}{\partial t} (\nabla \cdot \underline{A}) = \begin{cases} 0 & ; y > 0 , \\ \xi & ; y < 0 \end{cases} \quad (78)$$

Upon combining Eqs. (77a)(77b) and (76a), we have

$$\nabla^2\underline{A} - \mu_o\sigma \frac{\partial A}{\partial t} = \nabla\nabla\cdot A + \mu_o\sigma\nabla\phi - \mu_o\sigma(\underline{V} \times \underline{B}_o) \; ; y < 0 . \quad (79)$$

We now choose the Lorentz gauge [Eq. (67)] to obtain

$$\nabla^2\phi - \mu_o\sigma \frac{\partial\phi}{\partial t} = \xi \; ; y < 0 , \quad (80)$$

$$\nabla^2\underline{A} - \mu_o\sigma \frac{\partial A}{\partial t} = - \mu_o\sigma(\underline{V} \times \underline{B}_o) \; ; y < 0 . \quad (81)$$

These equations differ in form from their electrostatic and

magnetostatic counterparts only in the presence of terms $\mu_o\sigma \frac{\partial\phi}{\partial t}$ and $\mu_o\sigma \frac{\partial A}{\partial t}$, and in the fact that the electric field as given by Eq. (77b) contains the time varying additive term $- \frac{\partial A}{\partial t}$. These terms can be neglected for hydrodynamic effects with horizontal spatial wave numbers κ that are much greater than $\sqrt{\mu_o\sigma\omega}$ (ω - the temporal radian frequency). This is readily demonstrated by writing the solutions of Eqs. (80) and (81) as Fourier integrals with respect to the transverse (x,z) coordinates and time. For example, the solution of Eq. (81) can always be written in the form

$$\underline{A}(\underline{r},t) = \int_{-\infty}^{\infty} e^{i\omega t} d\omega \iint_{-\infty}^{\infty} e^{-i\underline{\kappa}\cdot\underline{\rho}}\, \underline{a}(y,\underline{\kappa},\omega)\, d^2\underline{\kappa} \tag{82}$$

where $\underline{a}$ satisfies

$$\frac{d^2}{dy^2}\,\underline{a} - (\kappa^2 + i\mu_o\sigma\omega)\,\underline{a} = \underline{f}(y,\underline{\kappa},\omega) \quad , \tag{83}$$

with $\underline{f}$ the Fourier transform of the right side of Eq. (81) with respect to t and $\underline{\rho}$. Clearly, if the source function $\underline{f}$ is significant only for

$$\kappa^2 >> \mu_o\sigma\omega \quad , \tag{84}$$

$\mu_o\sigma\omega$ on the left side of Eq. (83) may also be neglected, which amounts to dropping the time derivative in Eq. (81). The identical argument applies, of course, to Eq. (80). Moreover, under the same conditions (viz., Eq. 84), the time derivative in Eq. (77b) may also be neglected, so that $\underline{E} = -\nabla\phi$. To see this, we merely have to rewrite Eq. (77b) in terms of $\underline{A}$ with the aid of the Lorentz gauge. Thus, for $y < 0$

$$E = \frac{1}{\sigma\mu_0} [\nabla\nabla\cdot\underline{A} - \sigma\mu_0 \frac{\partial \underline{A}}{\partial t}] \quad . \qquad (85)$$

By representing $\underline{A}$ in terms of the Fourier transform as in Eq. (82), we again convince ourselves that for $\kappa^2 >> \mu_0\sigma\omega$, the time derivative in Eq. (85) may again be neglected. Under what conditions is Eq. (84) applicable? If we define the spatial wavelength of the hydrodynamic disturbance by $\lambda = \frac{2\pi}{\kappa}$, then with $\omega = 2\pi f$ the inequality in Eq. (84) reads

$$\lambda << \frac{10^3}{\sqrt{f}} \text{ meters} \quad . \qquad (86)$$

Thus, at frequencies as high as 1 Hz, this "short" wavelength approximation encompasses all wavelengths that are much less than 1 km. Clearly, Eq. (86) encompasses the range of predominant ocean surface wave phenomena. Also, for linear internal waves with frequencies as high as 10^{-2} Hz, Eq. (86) gives λ << 10 km.

The conclusions reached on the basis of the preceding heuristic arguments are fully supported by the results obtained with the aid of the exact formulation presented in Appendix D. Consequently, the dominant electric and magnetic field components induced by ocean currents that depend explicitly on time are given by the equations of electrostatics and magnetostatics in which the time variable enters simply as a parameter in the fluid velocity field.

IV. EXPLICIT EXPRESSIONS FOR THE MAGNETIC FIELD COMPONENTS UNDER THE QUASI-STATIC APPROXIMATION

Having established that electromagnetic fields induced by hydrodynamic phenomena with scale lengths much shorter than 1 km are governed by the equations of magnetostatics and electrostatics, we now proceed to obtain explicit expressions for the field components.

The induced magnetic fields follow by taking the curl of (69). The result can be written in the following form:

$$\underline{B}(\underline{r},t) \equiv \mu_0 \underline{H}(\underline{r},t)$$

$$= \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2\underline{\rho}' \;\; \underline{G}(\underline{r},\underline{r}')\, \underline{V}(\underline{r}',t)\,, \tag{87}$$

where $\underline{V}(\underline{r}',t)$ is a column matrix formed by the three fluid velocity components $V_x(\underline{r}',t)$, $V_y(\underline{r}',t)$, $V_z(\underline{r}',t)$ and the square matrix $\underline{G}(\underline{r},\underline{r}')$ is the hydrodynamic-magnetic Green's function. This matrix comprises the components of $\underline{G}$ in (70) and the components of the earth's magnetic field $\underline{B}_0$. Since the components of $\underline{G}$ are different for observation points above and below the water surface, we shall distinguish them by superscripts $+(y>0)$ and $-(y<0)$. After tedious but straightforward algebraic manipulations, one finds:

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$$G^{+}_{xx} = \sigma\mu_o B_{oy} \frac{\partial}{\partial x} G^{+}_{yx}(\underline{r},\underline{r}') \tag{88a}$$

$$G^{+}_{xy} = -\sigma\mu_o \underline{B}_o \cdot \nabla_T G^{+}_{yx}(\underline{r},\underline{r}') \tag{88b}$$

$$G^{+}_{xz} = \sigma\mu_o B_{oy} \frac{\partial}{\partial z} G^{+}_{yx}(\underline{r},\underline{r}') \tag{88c}$$

$$G^{\pm}_{yx} = -\sigma\mu_o B_{oy} \frac{\partial G_o(\underline{r},\underline{r}')}{\partial x} \tag{89a}$$

$$G^{\pm}_{yy} = \sigma\mu_o \underline{B}_o \cdot \nabla_T G_o(\underline{r},\underline{r}') \tag{89b}$$

$$G^{\pm}_{yz} = -\sigma\mu_o B_{oy} \frac{\partial G_o(\underline{r},\underline{r}')}{\partial z} \tag{89c}$$

$$G^{+}_{zx} = \sigma\mu_o B_{oy} \frac{\partial}{\partial x} G^{+}_{yz}(\underline{r},\underline{r}') \tag{90a}$$

$$G^{+}_{zy} = -\sigma\mu_o \underline{B}_o \cdot \nabla_T G^{+}_{yz}(\underline{r},\underline{r}') \tag{90b}$$

$$G^{+}_{zz} = \sigma\mu_o B_{oy} \frac{\partial}{\partial z} G^{+}_{yz}(\underline{r},\underline{r}') \tag{90c}$$

$$G^-_{xx} = \sigma\mu_o \left\{ B_{oy} \left[\frac{\partial}{\partial y} G_o(\underline{r},\underline{r}') - \frac{\partial}{\partial z} G^-_{yz}(\underline{r},\underline{r}') \right] + B_{oz} \frac{\partial}{\partial z} G^-_{yy}(\underline{r},\underline{r}') \right\} \quad (91a)$$

$$G^-_{xy} = \sigma\mu_o \left\{ B_{ox} \left[\frac{\partial}{\partial z} G^-_{yz}(\underline{r},\underline{r}') - \frac{\partial}{\partial y} G_o(\underline{r},\underline{r}') \right] - B_{oz} \frac{\partial}{\partial z} G^-_{yx}(\underline{r},\underline{r}') \right\} \quad (91b)$$

$$G^-_{xz} = \sigma\mu_o \left[B_{oy} \frac{\partial}{\partial z} G^-_{yx}(\underline{r},\underline{r}') - B_{ox} \frac{\partial}{\partial z} G^-_{yy}(\underline{r},\underline{r}') \right] \quad (91c)$$

$$G^-_{zx} = \sigma\mu_o \left[B_{oy} \frac{\partial}{\partial x} G^-_{yz}(\underline{r},\underline{r}') - B_{oz} \frac{\partial G^-_{yy}(\underline{r},\underline{r}')}{\partial x} \right] \quad (92a)$$

$$G^-_{zy} = \sigma\mu_o \left\{ B_{oz} \left[\frac{\partial}{\partial x} G^-_{yx}(\underline{r},\underline{r}') - \frac{\partial}{\partial y} G_o(\underline{r},\underline{r}') \right] - B_{ox} \frac{\partial}{\partial x} G^-_{yz}(\underline{r},\underline{r}') \right\} \quad (92b)$$

$$G^-_{zz} = \sigma\mu_o \left\{ B_{oy} \left[\frac{\partial G_o(\underline{r},\underline{r}')}{\partial y} - \frac{\partial G^-_{yx}(\underline{r},\underline{r}')}{\partial x} \right] + B_{ox} \frac{\partial}{\partial x} G^-_{yy}(\underline{r},\underline{r}') \right\} . \quad (92c)$$

There are two general classes of hydrodynamic disturbances of interest: spatially and temporally localized flows and wavelike disturbances, in which the time-dependent velocity field $\underline{V}$ is more conveniently expressed as a superposition integral of traveling waves of the form $\exp(-i\underline{k}_T \cdot \underline{\rho} + i\omega t)$. In the first class of problems, the volume integrals in (88) are best carried out directly; for wavelike hydrodynamic disturbances it is more convenient to initially express the components

of $\underline{\underline{G}}$ as two-dimensional Fourier transforms with respect to $\underline{\rho}$. For this purpose we define

$$G^{\pm}_{pq}(\underline{r},\underline{r}') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \hat{G}^{\pm}_{pq}(\underline{k}_T;y,y')\, e^{-i\underline{k}_T\cdot(\underline{\rho}-\underline{\rho}')}\, d^2\underline{k}_T \quad (93)$$

where p,q stand for x,y,z. Similarly for $\underline{V}(\underline{r}',t)$ we write

$$\underline{V}(\underline{r}',t) = \iint_{-\infty}^{\infty} \hat{\underline{V}}(\underline{k}_T,y',t)\, e^{-i\underline{k}_T\cdot\underline{\rho}'}\, d^2\underline{k}_T \; . \quad (94)$$

Forming a square matrix of the elements in (93), and denoting it by $\hat{\underline{\underline{G}}}(\underline{k}_T;y,y;)$ we have the equivalent form for (87) :

$$\underline{B}(\underline{r},t) = \iint_{-\infty}^{\infty} \hat{\underline{B}}(\underline{k}_T,y,t)\, e^{-i\underline{k}_T\cdot\underline{\rho}}\, d^2\underline{k}_T \quad , \quad (95)$$

with

$$\hat{\underline{B}}(\underline{k}_T,y,t) = \int_{-\infty}^{0} dy'\; \hat{\underline{\underline{G}}}\,(\underline{k}_T;y,y')\; \hat{\underline{V}}(\underline{k}_T,y',t) \quad . \quad (96)$$

The components of $\hat{\underline{G}}$ are found by taking the Fourier transforms of (88) through (92). These transforms are readily found from the corresponding transforms of G_o, G_{yx}, G_{yz}, G_{yy} as determined in Appendix C. There it is shown that

$$G^{\pm}_{yz} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2\underline{k}_T \, e^{-i\underline{k}_T \cdot (\underline{\rho}-\underline{\rho}')} \frac{(-ik_z) \, e^{-k_T|y\mp y'|}}{2k_T^2} , \quad (97a)$$

$$G^{\pm}_{yx} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2\underline{k}_T \, e^{-i\underline{k}_T \cdot (\underline{\rho}-\underline{\rho}')} \frac{(-ik_x) \, e^{-k_T|y\mp y'|}}{2k_T^2} , \quad (97b)$$

Also,

$$G_o = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2\underline{k}_T \, e^{-i\underline{k}_T \cdot (\underline{\rho}-\underline{\rho}')} \frac{e^{-k_T|y-y'|}}{2k_T} , \quad (97c)$$

$$G^{-}_{yy} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2\underline{k}_T \, e^{-i\underline{k}_T \cdot (\underline{\rho}-\underline{\rho}')} \frac{e^{-k_T|y-y'|} - e^{-k_T|y+y'|}}{2k_T} . \quad (97d)$$

Upon employing (97) in (88) - (92) one finds the following expressions for the elements of $\hat{\underline{G}}$:

$$\hat{G}^{+}_{xx} = -\sigma\mu_o \; B_{oy} \frac{k_x^2 \; e^{-k_T(y-y')}}{2k_T^2} \tag{98a}$$

$$\hat{G}^{+}_{xy} = \sigma\mu_o \left[\frac{k_x^2}{2k_T^2} B_{ox} + \frac{k_x k_z}{2k_T^2} B_{oz} \right] e^{-k_T(y-y')} \tag{98b}$$

$$\hat{G}^{+}_{xz} = -\sigma\mu_o \; B_{oy} \frac{k_x k_z}{2k_T^2} e^{-k_T(y-y')} \tag{98c}$$

$$\hat{G}^{\pm}_{yx} = i\sigma\mu_o \; B_{oy} \frac{k_x}{2k_T} \; e^{-k_T|y-y'|} \tag{99a}$$

$$\hat{G}^{\pm}_{yy} = -i\sigma\mu_o \left[\frac{k_x B_{ox} + k_z B_{oz}}{2k_T} \right] e^{-k_T|y-y'|} \tag{99b}$$

$$\hat{G}^{\pm}_{yz} = i\sigma\mu_o \; B_{oy} \frac{k_z}{2k_T} e^{-k_T|y-y'|} \tag{99c}$$

$$\hat{G}^{+}_{zx} = - \sigma\mu_o \; B_{oy} \frac{k_x k_z}{2k_T^2} \; e^{-k_T(y-y')} \tag{100a}$$

$$\hat{G}^+_{zy} = \sigma\mu_o \left(\frac{B_{ox}k_xk_z + B_{oz}k_z^2}{2k_T^2}\right) e^{-k_T(y-y')} \tag{100b}$$

$$\hat{G}^+_{zz} = -\sigma\mu_o \, B_{oy} \frac{k_T^2}{2k_T^2} \, e^{-k_T(y-y')} \tag{100c}$$

$$\hat{G}^-_{xx} =$$

$$\frac{\sigma\mu_o}{2}\begin{cases} \left(B_{oy} - \dfrac{ik_z}{k_T} B_{oz}\right)e^{-k_T(y'-y)} + \dfrac{k_z}{k_T}\left(\dfrac{k_z}{k_T} \; B_{oy} + i \, B_{oz}\right) e^{k_T(y+y')} \; ; \; y'>y \; , \\ -\left(B_{oy} + \dfrac{ik_z}{k_T} B_{oz}\right)e^{-k_T(y-y')} + \dfrac{k_z}{k_T}\left(\dfrac{k_z}{k_T} \; B_{oy} + i \, B_{oz}\right) e^{k_T(y+y')} \; ; \; y'<y \; , \end{cases} \tag{101a}$$

$$\hat{G}^-_{xy} =$$

$$\frac{\sigma\mu_o}{2}\begin{cases} \left(\dfrac{k_xk_z}{k_T^2} B_{oz} - \dfrac{k_z^2}{k_T^2} B_{ox}\right) e^{k_T(y+y')} - B_{ox} \, e^{-k_T(y'-y)} \; ; \; y'>y \; , \\ \left(\dfrac{k_xk_z}{k_T^2} B_{oz} - \dfrac{k_z^2}{k_T^2} B_{ox}\right) e^{k_T(y+y')} + B_{ox} \, e^{-k_T(y-y')} \; ; \; y'<y \; , \end{cases} \tag{101b}$$

$$\hat{G}^{-}_{xz} = -\frac{\sigma\mu_o k_z}{2\,k_T}\left[\left(\frac{k_x}{k_T}B_{oy} + iB_{ox}\right)e^{k_T(y+y')} - iB_{ox}\,e^{-k_T|y-y'|}\right], \qquad (101c)$$

$$\hat{G}^{-}_{zx} = -\frac{\sigma\mu_o k_x}{2\,k_T}\left[\left(\frac{k_z}{k_T}B_{oy} + iB_{oz}\right)e^{k_T(y+y')} - iB_{oz}\,e^{-k_T|y-y'|}\right], \qquad (102a)$$

$$\hat{G}^{-}_{zy} = \frac{\sigma\mu_o}{2}\begin{cases}\left(\dfrac{k_x k_z}{k_T^2}B_{ox} - \dfrac{k_x^2}{k_T^2}B_{oz}\right)e^{k_T(y+y')} - B_{oz}\,e^{-k_T(y'-y)}\,; & y'>y\ ,\\[2ex] \left(\dfrac{k_x k_z}{k_T^2}B_{ox} - \dfrac{k_x^2}{k_T^2}B_{oz}\right)e^{k_T(y+y')} + B_{oz}\,e^{-k_T(y-y')}\,; & y'<y\ ,\end{cases}$$

(102b)

$$\hat{G}^{-}_{zz} =$$

$$\frac{\sigma\mu_o}{2}\begin{cases}\left(B_{oy} - i\dfrac{k_x}{k_T}B_{ox}\right)e^{-k_T(y'-y)} + \dfrac{k_x}{k_T}\left(\dfrac{k_x}{k_T}B_{oy} + iB_{ox}\right)e^{k_T(y+y')}\,; & y'>y\ ,\\[2ex] -\left(B_{oy} + i\dfrac{k_x}{k_T}B_{ox}\right)e^{-k_T(y-y')} + \dfrac{k_x}{k_T}\left(\dfrac{k_x}{k_T}B_{oy} + iB_{ox}\right)e^{k_T(y+y')}\,; & y'<y\ .\end{cases}$$

(102c)

V. ELECTROMAGNETIC FIELDS INDUCED BY TRAVELING WAVE DISTURBANCES

The preceding formulas for the magnetic field components induced by hydrodynamic disturbances in deep ocean are generally valid. Their practical application is, however, limited to flow fields for which an adequate theoretical basis has been established. Here we shall single out and discuss only linear (small amplitude) internal waves and surface waves. The hydrodynamic background material is presented in Appendices A and E. We first consider internal waves.

A. MAGNETIC FIELDS INDUCED BY LINEAR INTERNAL WAVES

At frequencies well above the inertial frequency and in the absence of viscous effects, and furthermore provided the Väisälä frequency profile does not exhibit very abrupt changes with depth, the spatial Fourier transforms with respect to the transverse coordinates y,z of the internal wave velocity fields are given by Eqs. (E-26), (E-28) and (E-29) of Appendix E.

$$\hat{V}_x(\underline{k}_T,y) = -\frac{ik_x}{k_T^2}\sum_n \dot{\phi}_n(y)\left[A_n^+(\underline{k}_T)\, e^{i\Omega_n(k_T)t} + A_n^-(\underline{k}_T)\, e^{-i\Omega_n(k_T)t}\right], \quad (103a)$$

$$\hat{V}_y(\underline{k}_T,y) = \sum_n \phi_n(y)\left[A_n^+(\underline{k}_T)\, e^{i\Omega_n(k_T)t} + A_n^-(\underline{k}_T)\, e^{-i\Omega_n(k_T)t}\right], \quad (103b)$$

$$\hat{V}_z(\underline{k}_T,y) = -\frac{ik_z}{k_T^2}\sum_n \dot{\phi}_n(y)\left[A_n^+(\underline{k}_T)\, e^{i\Omega_n(k_T)t} + A_n^-(\underline{k}_T)\, e^{-i\Omega_n(k_T)t}\right], \quad (103c)$$

where we have replaced $\underline{K}$ employed in Appendix E by $\underline{k}_T$, in consonance with the notation in the preceding section.

Although the various quantities entering in (103) are defined in Appendix E, we repeat them here for ready reference. The ϕ_n are eigenfunctions of the internal wave modes satisfying the eigenvalue equation

$$\left[\frac{d^2}{dy^2} + k_T^2\left(\frac{N^2(y)}{\Omega_n^2} - 1\right)\right] \phi_n(y) = 0 \qquad (104)$$

with $\phi_n(0) = \phi_n(-D) = 0$, D being the ocean depth. For a deep ocean it is reasonable to assume that $D \to \infty$ in which case one of the boundary conditions should be replaced by $\lim_{y\to-\infty} \phi_n(y) \to 0$. A mathematically meaningful problem would require that this limit exist. This will be the case if the Väisälä frequency profile $N(y)$ is assumed to decrease continuously to zero past some depth as, for example, for the exponentially decreasing profile used by Garrett and Munk [8]. The $A_n^{\pm}(\underline{k}_T)$ are modal amplitudes which in general depend on the magnitude and direction of the transverse wave number; the dispersion relation for each mode is denoted by $\Omega_n(k_T)$, Ω_n being the angular frequency entering into (104).

The components of the induced magnetic field are obtained by substituting (103) into (96) and employing the defining relations for the matrix elements of $\hat{\underline{G}}$, e.g., in (98) - (102). For y>0 the three components of the induced magnetic field are

$$B_x(\underline{r},t) = \frac{\sigma\mu_o}{2}\iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}-k_T y}\, d^2\underline{k}_T \left[\left(\frac{k_x}{k_T}\right)\frac{\underline{k}_T}{k_T}\cdot\underline{B}_o - i\,\frac{k_x}{k_T}\,B_{oy}\right]$$

$$\cdot \sum_n \int_{-\infty}^{0} e^{k_T y'}\,\phi_n(y')dy' \left[A_n^{+}(\underline{k}_T)\,e^{i\Omega_n(k_T)t} + A_n^{-}(k_T)\,e^{-i\Omega_n(k_T)t}\right]$$

(105a)

$$B_y(\underline{r},t) = -\frac{\sigma\mu_o}{2}\iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}-k_T y}\, d^2\underline{k}_T \left(B_{oy} + i\,\frac{\underline{k}_T}{k_T}\cdot\underline{B}_o\right)$$

$$\cdot \sum_n \int_{-\infty}^{0} e^{k_T y'}\,\phi_n(y')dy' \left[A_n^+(\underline{k}_T)\, e^{i\Omega_n(k_T)t} + A_n^-(\underline{k}_T)\, e^{-i\Omega_n(k_T)t}\right] \tag{105b}$$

$$B_z(\underline{r},t) = \frac{\sigma\mu_o}{2}\iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}-k_T y}\, d^2\underline{k}_T \left[\left(\frac{k_z}{k_T}\right)\frac{\underline{k}_T}{k_T}\cdot\underline{B}_o - i\,B_{oy}\frac{k_z}{k_T}\right]$$

$$\cdot \sum_n \int_{-\infty}^{0} e^{k_T y'}\,\phi_n(y')dy' \left[A_n^+(\underline{k}_T)\, e^{i\Omega_n(k_T)t} + A_n^-(\underline{k}_T)\, e^{-i\Omega(k_T)t}\right]. \tag{105c}$$

In the derivation of these equations use has been made of the fact that $\phi_n(y)$ vanishes at $y = 0$ and $y = -\infty$, so that

$$\int_{-\infty}^{0} \dot{\phi}_n(y')\, e^{k_T y'}\, dy' = -\, k_T \int_{-\infty}^{0} \phi_n(y')\, e^{k_T y'}\, dy' \quad .$$

For $y<0$ the expressions for the fields are somewhat more complicated. One finds

$$B_x(\underline{r},t) = \frac{\sigma\mu_o}{2}\iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}}\, d^2\underline{k}_T \;.$$

$$\sum_n \left\{ B_{oy}\left[\frac{2ik_x}{k_T^2}\phi_n(y) - \frac{ik_x}{k_T}\left(e^{-k_T y}\int_{-\infty}^{y} e^{k_T y'}\phi_n(y')dy' + e^{k_T y}\int_y^0 e^{-k_T y'}\phi_n(y')dy'\right)\right]\right.$$

$$\left. + \frac{k_x}{k_T}\left(\underline{B}_o\cdot\frac{\underline{k}_T}{k_T}\right)\left(e^{-k_T y}\int_{-\infty}^{y} e^{k_T y'}\phi_n(y')dy' - e^{k_T y}\int_y^0 e^{-k_T y'}\phi_n(y')dy'\right)\right\}$$

$$\cdot\left\{A_n^+(\underline{k}_T)\, e^{i\Omega_n(k_T)t} + A_n^-(\underline{k}_T)\, e^{-i\Omega_n(k_T)t}\right\} \qquad (106a)$$

$$B_z(\underline{r},t) = \frac{\sigma\mu_o}{2}\iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}}\, d^2\underline{K}_T \;.$$

$$\sum_n \left\{ B_{oy}\left[\frac{2ik_z}{k_T^2}\phi_n(y) - \frac{ik_z}{k_T}\left(e^{-k_T y}\int_{-\infty}^{y} e^{k_T y'}\phi_n(y')dy' + e^{k_T y}\int_y^0 e^{-k_T y'}\phi_n(y')dy'\right)\right]\right.$$

$$\left. + \frac{k_z}{k_T}\left(\underline{B}_o\cdot\frac{\underline{k}_T}{k_T}\right)\left(e^{-k_T y}\int_{-\infty}^{y} e^{k_T y'}\phi_n(y')dy' - e^{k_T y}\int_y^0 e^{-k_T y'}\phi_n(y')dy'\right)\right\} .$$

$$\cdot\left\{A_n^+(\underline{k}_T)\, e^{i\Omega_n(k_T)t} + A_n^-(k_T)\, e^{-i\Omega_n(k_T)t}\right\} \qquad (106b)$$

$$B_y(\underline{r},t) = \frac{\sigma\mu_o}{2}\iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}}\, d^2\underline{k}_T$$

$$\cdot \sum_n \Bigg\{ B_{oy}\left[e^{k_T y}\int_y^0 dy'\, e^{-k_T y'}\, \phi_n(y')dy' - e^{-k_T y}\int_{-\infty}^{y} dy'\, e^{k_T y'}\, \phi_n(y')dy' \right]$$

$$- i\,\frac{\underline{k}_T\cdot\underline{B}_o}{k_T}\left[e^{-k_T y}\int_{-\infty}^{y} \phi_n(y')\, e^{k_T y'}\, dy' + e^{k_T y}\int_y^0 \phi_n(y')\, e^{-k_T y'}\, dy' \right]\Bigg\}$$

$$\cdot \left\{ A_n^+(\underline{k}_T)\, e^{i\Omega_n t} + A_n^-(\underline{k}_T)\, e^{-i\Omega_n t} \right\} \tag{106c}$$

The formulas for the induced field components above the ocean surface, Eqs. (105a - 105c), can be written in a more compact form. For this purpose we introduce the complex unit vector $\underline{a}$,

$$\underline{a} = \frac{1}{\sqrt{2}}\left(\frac{\underline{k}_T}{k_T} - i\underline{y}_o\right) \ . \tag{107}$$

Evidently $\underline{a}\cdot\underline{a}^* = 1$. One then finds that for $y > 0$, the spatial Fourier transform of the induced magnetic field components is

$$\hat{\underline{B}}(\underline{k}_T,y,t) =$$

$$\sigma\mu_o\, e^{-k_T y}\,\underline{a}(\underline{a}\cdot\underline{B}_o)\sum_n \int_{-\infty}^{0} \phi_n(y')\, e^{k_T y'}\, dy' \left[A_n^+\, e^{i\Omega_n t} + A_n^-\, e^{-i\Omega_n t}\right] , \tag{108}$$

while the induced field itself is given by

$$\underline{B}(\underline{r},t) = \iint_{-\infty}^{\infty} \hat{\underline{B}}(\underline{k}_T,y,t)\, e^{-i\underline{k}_T\cdot\underline{\rho}}\, d^2\underline{k}_T \quad . \tag{109}$$

The relationship among the three components of the induced field is best illustrated by resolving its spatial transform $\hat{\underline{B}}$ along the three mutually perpendicular unit vectors $\frac{\underline{k}_T}{K_T}$, $\underline{t}$, $\underline{y}_o$, forming a right-handed cartesian coordinate system for each wavenumber $\underline{k}_T$. The geometrical relationship is illustrated in Fig. 1.

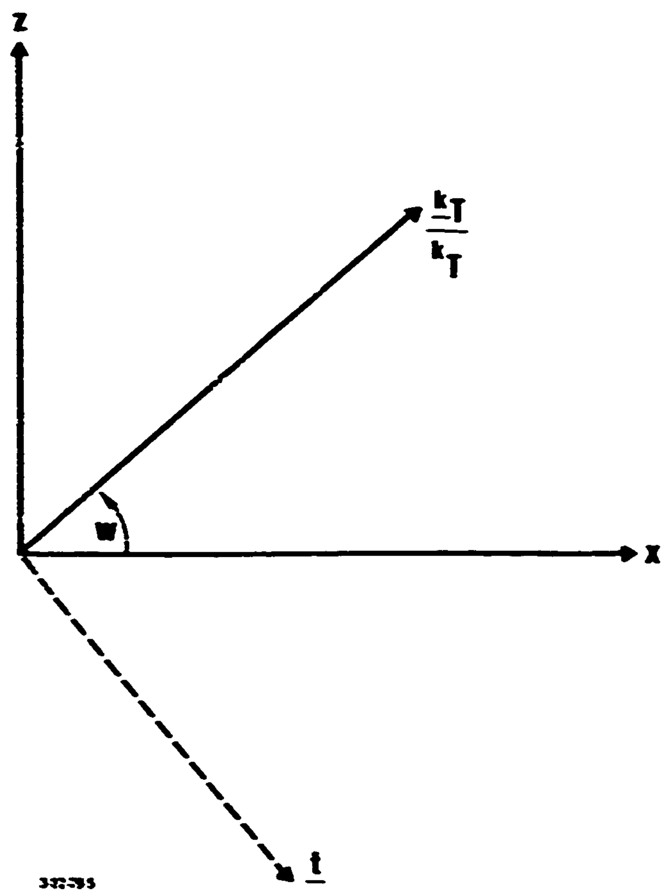

FIGURE 1.

The unit vector $\underline{t}$ points along the wave crest and normal to the direction of propagation $\frac{\underline{k}_T}{k_T}$ of the internal wave field. When resolved along these three unit vectors, $\hat{\underline{B}}$ may be written as follows:

$$\hat{\underline{B}} = \hat{B}_k \left(\frac{\underline{k}_T}{k_T} \right) + \hat{B}_t \, \underline{t} + \hat{B}_y \, \underline{y}_o \quad . \tag{110}$$

Since $\frac{\underline{k}_T}{k_T} x \underline{y}_o = \underline{t}$, one finds

$$\hat{B}_t = 0 \quad , \tag{111a}$$

$$\hat{B}_k = \frac{\sigma\mu_o}{\sqrt{2}} \, e^{-k_T y} \, (\underline{a}\cdot\underline{B}_o) \sum_n \int_{-\infty}^{0} \phi_n(y') \, e^{k_T y} \, dy' \left[A_n^{+} \, e^{i\Omega_n t} + A_n^{-} \, e^{-i\Omega_n t} \right], \tag{111b}$$

$$\hat{B}_y = - i\hat{B}_k \quad . \tag{111c}$$

Thus, each Fourier component of the spatial transform $\hat{\underline{B}}$ lies entirely in the plane formed by the vertical and the propagation vector $\underline{k}_T$. Moreover, the vertical component of $\hat{\underline{B}}$ is equal in magnitude to the component along the direction of propagation and 90° out of time phase. If the internal wave field is unidirectional, i.e., comprising only a single traveling wave, the preceding observations apply to the induced field itself. In that case, the vertical and horizontal components of the induced magnetic field may be considered as forming a circularly polarized field, an observation that has also been made by Podney [7].

Thus far, we have focused entirely on the components of the induced magnetic field. The most sensitive magnetic detection instruments are superconducting gradiometers which measure,

to a good approximation, spatial derivatives of the magnetic field components. In order to retain maximum generality, we shall define the gradient of the magnetic field relative to any two nonparallel unit vectors $\underline{\ell}_p$, $\underline{\ell}_q$, $\underline{\ell}_p \cdot \underline{\ell}_p = 1$, $\underline{\ell}_q \cdot \underline{\ell}_q = 1$. The magnetic field gradient with respect to direction $\underline{\ell}_p$ of the induced magnetic field along the direction $\underline{\ell}_q$ will be denoted by G_{pq}. Thus,

$$G_{pq}(\underline{r},t) = \underline{\ell}_p \cdot \nabla(\underline{B} \cdot \underline{\ell}_q) \; . \tag{112}$$

Expressed in terms of the spatial Fourier transform $\hat{G}_{pq}(\underline{k}_T,y,t)$,

$$G_{pq}(\underline{r},t) = \iint_{-\infty}^{\infty} e^{-i\underline{k}_T \cdot \underline{\rho}} \; \hat{G}_{pq}(\underline{k}_T,y,t) \, d^2\underline{k}_T \; . \tag{113}$$

Upon applying formula (112) to (108) one obtains

$$\hat{G}_{pq}(\underline{k}_T,y,t) =$$

$$- i\sigma\mu_o \sqrt{2}\, k_T \, e^{-k_T y} (\underline{\ell}_p \cdot \underline{a})(\underline{\ell}_q \cdot \underline{a})(\underline{a} \cdot \underline{B}_o) \sum_n \int_{-\infty}^{0} \phi_n(y') \, e^{k_T y'} dy' \left[A_n^+ e^{i\Omega_n t} + A_n^- e^{-i\Omega_n t} \right] . \tag{114}$$

Note that $\hat{G}_{pq} = \hat{G}_{qp}$, which is a direct consequence of the fact that in the quasi-static approximation employed herein $\nabla \times \underline{B} = 0$ above the water surface.

Since only the relative orientation between the geomagnetic field and the components of the induced gradients (or field compo- ents) is important, we may assume, without loss of generality, that the geomagnetic field lies in the xy plane, with the x-axis pointing in the direction of magnetic south. As usual we denote

the dip angle by ϕ_D so that

$$B_{ox} = B_o \cos \phi_D \quad , \tag{115a}$$

$$B_{oy} = B_o \sin \phi_D \quad , \tag{115b}$$

$$B_{oz} = 0 \quad . \tag{115c}$$

The total geomagnetic field B_o may be written in terms of Λ, the magnetic latitude, as follows [7]:

$$B_o = \frac{B_p}{2} (1 + 3 \sin^2 \Lambda)^{1/2} \quad , \tag{116}$$

where $\Lambda = \frac{\pi}{2}$ in the polar region and $\Lambda = 0$ in the equatorial region. The numerical value of B_p in MKS units is

$$B_p = 6.24 \times 10^{-5} \frac{\text{webers}}{m^2} \text{ (or Tesla)} = 6.24 \times 10^7 \text{ pT} \quad . \tag{117}$$

We shall also need the relation between Λ and the dip angle, which is

$$\tan \phi_L = 2 \tan \Lambda \tag{118}$$

The total geomagnetic field B_o may then be written in terms of ϕ_D, as follows

$$B_o = \frac{B_p}{(1 + 3 \cos^2 \phi_D)^{1/2}} \text{ , } 0 \leq \phi_D \leq \pi/2 \quad . \tag{119}$$

With the geomagnetic field lying in the xy plane we now choose three mutually orthogonal unit vectors which we denote by

$\underline{\ell}_1$, $\underline{\ell}_2$, $\underline{\ell}_3$ and which form a right-handed cartesian system ($\underline{\ell}_1 \times \underline{\ell}_2 = \underline{\ell}_3$) ; $\underline{\ell}_1$ and $\underline{\ell}_3$ lie in the xz plane while $\underline{\ell}_2 \equiv \underline{y}_o$, as shown in Fig. 2. The arbitrary angle

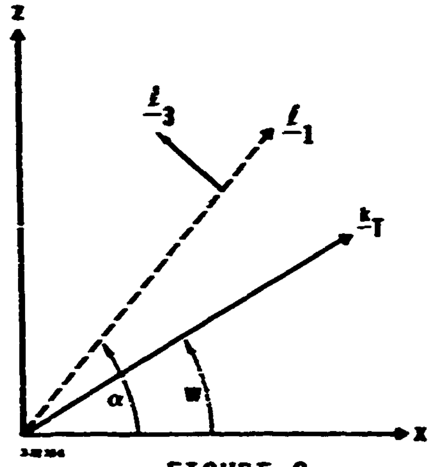

FIGURE 2.

between $\underline{\ell}_1$ and the xy plane is denoted by α. Also shown is the (horizontal) wave propagation vector $\underline{k}_T$, whose angle with the x-axis we denote by w. The induced magnetic fields and gradients will be expressed relative to $\underline{\ell}_1$, $\underline{\ell}_2$, $\underline{\ell}_3$; the relative orientation of these unit vectors and the geomagnetic field is shown in Fig. 3.

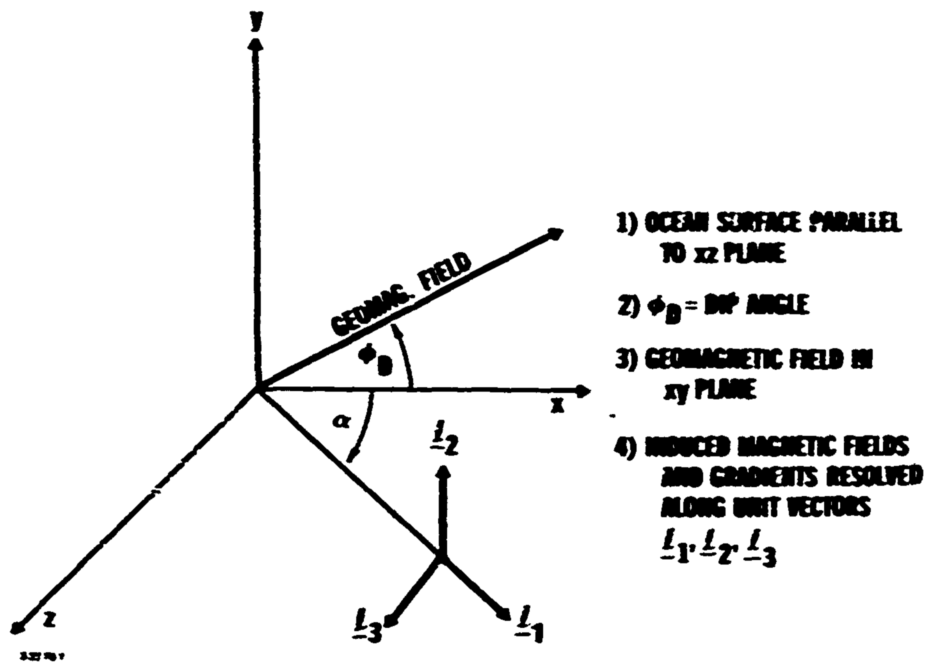

FIGURE 3.

From Fig. 2 one finds that the projections on $\underline{\ell}_1$, $\underline{\ell}_2$ and $\underline{\ell}_3$ of the complex unit vector defined in (107) are given by

$$\underline{\ell}_1 \cdot \underline{a} = \frac{1}{\sqrt{2}} \cos(w-\alpha) \quad , \tag{120a}$$

$$\underline{\ell}_2 \cdot \underline{a} = -\frac{i}{\sqrt{2}} \quad , \tag{120b}$$

$$\underline{\ell}_3 \cdot \underline{a} = \frac{1}{\sqrt{2}} \sin(w-\alpha) \quad . \tag{120c}$$

With the aid of (115) and (119) we also find the projection of $\underline{a}$ on $\underline{B}_o$

$$\underline{a} \cdot \underline{B}_o = \frac{B_o}{\sqrt{2}} (\cos w \cos \phi_D - i\sin \phi_D)$$

$$= \frac{B_D/\sqrt{2}}{(1 + 3\cos^2 \phi_D)^{1/2}} (\cos \phi_D \cos w - i\sin \phi_D) , \tag{121}$$

where we have employed the polar form for $\underline{k}_T$:

$$k_x = k_T \cos w \quad ,$$

$$k_z = k_T \sin w \quad .$$

With the aid of the preceding relations, the three components of the Fourier transform of magnetic field gradient for $p \neq q$ in Eq. (114) become

$$\hat{G}_{12}(\underline{k}_T,y,t) = \frac{-\sigma\mu_o B_p \cos(\kappa-\alpha)[\cos\phi_D \cos\kappa - i\sin\phi_D]}{2(1+3\cos^2\phi_D)^{1/2}}\, k_T\, e^{-k_T y}\, h(\underline{k}_T,t), \quad (122a)$$

$$\hat{G}_{13}(\underline{k}_T,y,t) = \frac{-\sigma\mu_o B_p \sin 2(\kappa-\alpha)[\cos\phi_D \cos\kappa - i\sin\phi_D]}{4(1+3\cos^2\phi_D)^{1/2}}\, k_T\, e^{-k_T y}\, h(\underline{k}_T,t), \quad (122b)$$

$$\hat{G}_{23}(\underline{k}_T,y,t) = \frac{-\sigma\mu_o B_p \sin(\kappa-c)[\cos\phi_D \cos\kappa - i\sin\phi_D]}{2(1+3\cos^2\phi_D)^{1/2}}\, k_T\, e^{-k_T y}\, h(\underline{k}_T,t), \quad (122c)$$

where $h(\underline{k}_T,t)$ depends only on the hydrodynamic aspects of the internal wave field, and is given by

$$h(\underline{k}_T,t) = \sum_n \int_{-\infty}^{0} \phi_n(y')\, e^{k_T y'}\, dy' \left[A_n^+(\underline{k}_T)\, e^{i\Omega_n t} + A_n^-(\underline{k}_T)\, e^{-i\Omega_n t} \right] . \quad (123)$$

From (122) we observe that for a unidirectional internal wave spectrum ($\kappa = \alpha$) $\hat{G}_{13} = \hat{G}_{23} \equiv 0$ so that only $\hat{G}_{12}$, the gradient of the vertical field component with respect to the wave propagation direction*, contributes. This is in accord with the observation made previously that a unidirectional internal wave induces no fields along the wave crest, i.e., the induced magnetic fields form a strictly two-dimensional pattern with no variation orthogonal to the plane formed by the vertical and the wave propagation vector.

For future reference we also resolve the spatial Fourier transform of the induced magnetic field $\hat{\underline{B}}$ along the three orthogonal vector $\underline{\ell}_1$, $\underline{\ell}_2$, $\underline{\ell}_3$. We employ the notation $\hat{B}_1$, $\hat{B}_2$ and $\hat{B}_3$. Thus, with the aid of (121) (107) and (120) one finds

$$\hat{B}_1(\underline{k}_T,y,t) = \sigma\mu_o B_p \frac{\cos(\kappa-\alpha)[\cos\phi_D \cos\kappa - i\sin\phi_D]}{2(1+3\cos^2\phi_D)^{1/2}}\, e^{-k_T y}\, h(\underline{k}_T,t) \quad , \quad (124a)$$

*which, by symmetry, is identical to the gradient with respect to y of the field component along the wave propagation direction.

$$\hat{B}_2(\underline{k}_T,y,t) = -i\sigma\mu_o B_p \frac{\cos\phi_D \cos w - i\sin\phi_D}{2(1+3\cos^2\phi_D)^{1/2}} e^{-k_T y} h(\underline{k}_T,t) \quad , \tag{124b}$$

$$\hat{B}_3(\underline{k}_T,y,t) = \sigma\mu_o B_p \frac{\sin(w-\alpha)[\cos\phi_D \cos w - i\sin\phi_D]}{2(1+3\cos^2\phi_D)^{1/2}} e^{-k_T y} h(\underline{k}_T,t) \quad , \tag{124c}$$

where again $\underline{h}(\underline{k}_T,t)$ is given by (123). Note that the essential difference between the spatial Fourier transforms of the gradients in (122) and the Fourier transforms of the field components in (124) is that the former comprise the additional multiplicative factor k_T, a direct consequence of the differentiation operation along the horizontal coordinates. Indeed, we find the following relations between (124) and (122):

$$\hat{G}_{12}(\underline{k}_T,y,t) = -\, k_T\, \hat{B}_1(\underline{k}_T,y,t) \quad , \tag{125a}$$

$$\hat{G}_{23}(\underline{k}_T,y,t) = -\, k_T\, \hat{B}_3(\underline{k}_T,y,t) \quad , \tag{125b}$$

$$\hat{G}_{13}(\underline{k}_T,y,t) = \frac{k_T}{2} \sin 2(w-\alpha)\, \hat{B}_2(\underline{k}_T,y,t) . \tag{125c}$$

Clearly, for a unidirectional internal wave field ($w = \alpha$), the gradient is obtained from the horizontal component of the magnetic field through a multiplication by the negative of the wave number. Also, for a more general wave number spectrum, the presence of k_T as a multiplicative factor will tend to weigh more heavily the short wavelength portion of the internal wave spectrum. This, of course, is hardly surprising since the gradients are proportional to a derivative of the field component with respect to the horizontal direction.

We shall postpone the discussion of the application of these formulas to the computation of the spectra of the induced magnetic fields and their gradients until Chapter VI. At present, we turn to the development of similar formulas for surface wave induced magnetic fields.

B. MAGNETIC FIELDS INDUCED BY SURFACE WAVES

The spatial Fourier transform of the velocity field associated with small amplitude surface waves in deep ocean, as given by Eq. (24) in Appendix A, reads

$$\hat{\underline{V}}(\underline{k}_T,y,t) = e^{-k_T y}\,\Omega k_T^{-1}\left[A^+(\underline{k}_T)\,e^{i\Omega t} - A^-(\underline{k}_T)\,e^{-i\Omega t}\right](\underline{k}_T + i\underline{y}_o\,k_T)\ , \quad (126)$$

where Ω is given by the dispersion relationship given in (A-20) as

$$\Omega = +\sqrt{k_T g}\ . \quad (127)$$

We shall concern ourselves here only with induced fields above the ocean surface.

With (126) substituted in (96) one finds for $y > 0$

$$\hat{B}_x = \frac{\sigma\mu_o}{4}\,e^{-k_T y}\left(\frac{k_x}{k_T}\right)\left[\frac{\underline{k}_T}{k_T}\cdot\underline{B}_o + iB_{oy}\right] h_s(\underline{k}_T,t)\ , \quad (128a)$$

$$\hat{B}_y = \frac{\sigma\mu_o}{4}\,e^{-k_T y}\left[B_{oy} - i\,\frac{\underline{k}_T}{k_T}\cdot\underline{B}_o\right] h_s(\underline{k}_T,t)\ , \quad (128b)$$

$$\hat{B}_z = \frac{\sigma\mu_o}{4}\,e^{-k_T y}\left(\frac{k_z}{k_T}\right)\left[\frac{\underline{k}_T}{k_T}\cdot\underline{B}_o + iB_{oy}\right] h_s(\underline{k}_T,t)\ , \quad (128c)$$

where

$$h_s(\underline{k}_T,t) = \frac{i\Omega}{k_T}\left[A^+(\underline{k}_T)\,e^{i\Omega t} - A^-(\underline{k}_T)\,e^{-i\Omega t}\right]\ . \quad (129)$$

Eq. (128) can also be written in terms of the unit projection vector $\underline{a}$ in (107):

$$\hat{\underline{B}}(\underline{k}_T,y,t) = \frac{\sigma\mu_o}{2} e^{-k_T y} \underline{a}(\underline{a}^*\cdot\underline{B}_o) h_s(\underline{k}_T,t) \quad . \tag{130}$$

The algebraic form is quite similar to that obtained for internal waves in (108). When $\hat{\underline{B}}$ is resolved along $\underline{\ell}_1$, $\underline{\ell}_2$, $\underline{\ell}_3$ in Fig. 3, one obtains expressions analogous to those in (124):

$$\hat{B}_1(\underline{k}_T,y,t) = \frac{\sigma\mu_o}{4} B_p \frac{\cos(w-\alpha)[\cos\phi_D \cos w + i\sin\phi_D]}{(1+3\cos^2\phi_D)^{1/2}} e^{-k_T y} h_s(\underline{k}_T,t), \tag{131a}$$

$$\hat{B}_2(\underline{k}_T,y,t) = -i\frac{\sigma\mu_o}{4} B_p \frac{\cos\phi_D \cos w + i\sin\phi_D}{(1+3\cos^2\phi_D)^{1/2}} e^{-k_T y} h_s(\underline{k}_T,t) \quad , \tag{131b}$$

$$\hat{B}_3(\underline{k}_T,y,t) = \frac{\sigma\mu_o}{4} B_p \frac{\sin(w-\alpha)[\cos\phi_D \cos w + i\sin\phi_D]}{(1+3\cos^2\phi_D)^{1/2}} e^{-k_T y} h_s(\underline{k}_T,t) \, . \tag{131c}$$

For a unidirectional surface wave we may set $w = \alpha$. One then observes that the induced magnetic field lies entirely in the plane containing the vertical and the wave propagation direction $\underline{k}_T$; the vertical and the horizontal component of the induced fields again are equal in magnitude and 90 deg out of time phase, just as for unidirectional internal waves. The Fourier transform of the induced magnetic field gradient above the ocean surface reads

$$\hat{G}_{pq}(\underline{k}_T,y,t) = -\frac{i\sigma\mu_o\sqrt{2}}{2} k_T(\underline{\ell}_p\cdot\underline{a})(\underline{\ell}_q\cdot\underline{a}) \, \underline{a}^*\cdot\underline{B}_o \, e^{-k_T y} h_s(\underline{k}_T,t) \quad . \tag{132}$$

When resolved along the mutually perpendicular directions $\underline{\ell}_1$, $\underline{\ell}_2$, $\underline{\ell}_3$ in Fig. 3, the three gradients $\hat{G}_{11}$, $\hat{G}_{12}$, $\hat{G}_{13}$ are again related to the $\hat{B}_1$ $\hat{B}_2$ $\hat{B}_3$ in (131) by Eq. (125). Thus, above the surface the geometrical and phase relationships among the induced magnetic fields and gradients for unidirectional surface waves and unidirectional internal waves are identical. Of course, the distribution of energy in frequency and wave number space in the two cases are, in general, quite different.

C. ELECTRIC FIELDS INDUCED BY SURFACE WAVES

Surface waves also induce electric fields above the ocean surface. On the other hand, to the extent that Coriolis effects can be neglected, the electric field above the ocean surface arising from linear internal waves is identically zero. The last statement follows from Eq. (59a), which gives the electrostatic potential induced by a velocity field with zero normal velocity at the ocean surface. Since this corresponds to the boundary condition for internal waves, no electric field can be induced for $y > 0$ if the normal component of vorticity is everywhere zero. That the vertical component of vorticity vanishes may be verified directly from (103).

The electric field induced by surface wave motion follows from (20). We first express the free space Green's function in the integrand with the aid of (97c) and express $V_z(x',0,z',t)$, $V_x(x',0,z',t)$ in terms of their spatial Fourier transforms $\hat{V}_z(\underline{k}_T,0,t)$, $\hat{V}_x(\underline{k}_T,0,t)$. This yields

$$\phi(\underline{\rho},y,t) = \iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho} - k_T y} \frac{\hat{V}_z(\underline{k}_T,0,t)\, B_{ox} - \hat{V}_x(\underline{k}_T,0,t)\, B_{oz}}{k_T} d^2\underline{k}_T \quad . \tag{133}$$

Upon taking the negative gradient and substituting for $\hat{V}_z$, $\hat{V}_x$ from (126) one obtains

$$\underline{E}(\underline{\rho},y,t) = -\nabla\phi = \iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}}\, \hat{\underline{E}}(\underline{k}_T,y,t)\, d^2\underline{k}_T \quad , \tag{134}$$

where

$$\hat{\underline{E}}(\underline{k}_T,y,t) = \sqrt{2}\ \underline{a}\left[\left(\underline{y}_c \times \underline{k}_T\right)\cdot \underline{B}_o\right] e^{-k_T y}\ h_s(\underline{k}_T,t)\ . \qquad (135)$$

We now resolve $\hat{\underline{E}}$ along $\underline{\ell}_1\ \underline{\ell}_2\ \underline{\ell}_3$ in Fig. 3 to obtain

$$\hat{E}_1(\underline{k}_T,y,t) = B_p \frac{\cos\phi_D}{(1+3\cos^2\phi_D)^{1/2}} \cos(w-\alpha)\ \sin w\left(k_T\ e^{-k_T y}\right) h_s(\underline{k}_T,t)\ , \qquad (136)$$

$$\hat{E}_2(\underline{k}_T,y,t) = -iB_p \frac{\cos\phi_D}{(1+3\cos^2\phi_D)^{1/2}} \sin w\left(k_T\ e^{-k_T y}\right) h_s(\underline{k}_T,t)\ , \qquad (137)$$

$$\hat{E}_3(\underline{k}_T,y,t) = B_p \frac{\cos\phi_D}{(1+3\cos^2\phi_D)^{1/2}} \sin(w-\alpha)\ \sin w\left(k_T\ e^{-k_T y}\right) h_s(\underline{k}_T,t)\ . \qquad (138)$$

Suppose we again consider a unidirectional surface wave, i.e., set $w = \alpha$. Then, just as was the case for the magnetic field, the electric field lies entirely in the plane of the wave propagation vector and the vertical. We again observe that the characteristic 90-deg phase relation obtains between the two equal amplitude orthogonal components $\hat{E}_1$ and $\hat{E}_2$. As the direction of propagation is varied, the electric field attains a maximum at $w = \alpha = \pi/2$ (normal to the plane containing the geomagnetic field) and vanishes at $w = \alpha = 0$, i.e., when the surface wave propagates in the direction of the geomagnetic field. Note also that only the horizontal component of the geomagnetic field is responsible for inducing an electric field: when the dip angle is 90 deg, all electric field components vanish. One curious fact, which has already been remarked in Chapter II, is that the electric field appears to be independent of the conductivity σ. This independence is only approximate and holds only if the conductivity is sufficiently high, i.e., when the nonlinear terms on the right of (13) are neglected.

D. PROPAGATION OF TRAVELING WAVE-INDUCED ELECTROMAGNETIC FIELDS ABOVE THE OCEAN SURFACE

Although the preceding formulas give correctly the dominant field components above the ocean surface, they fail to describe the propagation of electromagnetic energy. For example, for internal waves, only a time-varying magnetic field--but no electric field--is induced above the ocean surface. Under these conditions, the Poynting vector above the ocean surface is identically zero, with the implication that no electromagnetic power is coupled from an internal wave to the region above the ocean surface. If this were really the case then such a time-varying field could never be detected, since any detection process must necessarily be accompanied by the extraction of a finite amount of power. The electric field component that would account for such power extraction is evidently set equal to zero once the quasi-static approximation is employed. Even though this field component is "small", it must be large enough so that a product of the form $H_y E$ x constant yields a detectable power level. This constant can be nothing else but a suitably normalized electromagnetic wave admittance. We shall presently find that if the magnetic field is induced by a single mode internal wave, the wave admittance is given by $\sqrt{\epsilon_o/\mu_o}\ c/v_p$, where c is the speed of light in vacuo, and v_p is the phase velocity of the internal wave; the electric field component E entering into the product EH_y x constant is parallel to the ocean surface and orthogonal to the horizontal propagation vector of the internal wave. The electromagnetic power transfer above the ocean surface takes place in the direction parallel to the direction of propagation of the internal wave. This electromagnetic power is transported along the ocean surface with the phase velocity v_p. Structurally, we obtain an H-mode wave, since it is characterized by a magnetic field component along the direction of propagation. On the other hand, a hydrodynamic surface wave will be found to generate two types of electromagnetic surface

waves: an H-mode wave, and an E-mode wave. The magnetic field component of the latter is normal to the ocean surface, and is neglected in the quasi-static approximation. The two electric field components are retained, and are given by the formulas in the preceding section.

We shall employ results from the exact formulation for electromagnetic fields induced by general time-varying hydrodynamic disturbances as presented in Appendix D. In the following, we first present a detailed discussion for surface waves. The structure of internal-wave-generated electromagnetic fields then follows almost by inspection.

For simplicity, consider a single-frequency unidirectional hydrodynamic surface wave of amplitude A, propagating in the direction $w = \alpha$ (Fig. 3) with wave number $\underline{K}$. As shown in Appendix D, the electromagnetic fields above the ocean surface can be represented as the sum of two *electromagnetic surface waves*; one designated as an E-mode (TM mode) wave (no H-field in the direction of propagation), the other designated as an H-mode (TE mode) wave (no E-field in the direction of propagation). When subjected to the approximations

$$\frac{\Omega\mu_o\sigma}{K^2} \ll 1$$

$$\left|\frac{\Omega\varepsilon_o}{\sigma}\sqrt{1 + \frac{i\Omega\mu_o\sigma}{K^2}}\right| \ll 1 \quad ,$$

the fields of these electromagnetic surface waves are given by Eqs. (D-113) and (D-114). These approximations are, of course, implicit also in the quasi-static approach [cf. Eq.(86)]. The latter, however, encompasses the *additional approximation*

of neglecting the displacement current and magnetic induction effects above the ocean surface, both of which effects we presently include. With the aid of (115) and (119), the electromagnetic field components of the E and H-mode assume the following form:

(1) E-modes:

$$E_2(x,y,z,t) = \frac{A\Omega\, B_p \cos\phi_D \sin\kappa}{(1 + 3\cos^2\phi_D)^{1/2}}\, e^{-Ky - i\underline{K}\cdot\underline{\rho} + i\Omega t}, \tag{139a}$$

$$H_3(x,y,z,t) = \left(\frac{\Omega\varepsilon_o}{K}\right) \frac{A\Omega\, B_p \cos\phi_D \sin\kappa}{(1 + 3\cos^2\phi_D)^{1/2}}\, e^{-Ky - i\underline{K}\cdot\underline{\rho} + i\Omega t}, \tag{139b}$$

$$H_1(x,y,z,t) = i\, \frac{A\Omega\, B_p \cos\phi_D \sin\kappa}{(1 + 3\cos^2\phi_D)^{1/2}}\, e^{-Ky - i\underline{K}\cdot\underline{\rho} + i\Omega t}, \tag{139c}$$

(2) H-modes:

$$H_2(x,y,z,t) = \frac{\sigma A\Omega\, B_p[\cos\phi_D \cos\kappa + i\sin\phi_D]}{4K(1 + 3\cos^2\phi_D)^{1/2}}\, e^{-Ky - i\underline{K}\cdot\underline{\rho} + i\Omega t}, \tag{140a}$$

$$E_3(x,y,z,t) = -\left(\frac{\Omega\mu_o}{K}\right) \frac{\sigma A\Omega\, B_p[\cos\phi_D \cos\kappa + i\sin\phi_D]}{4K(1 + 3\cos^2\phi_D)^{1/2}}\, e^{-Ky - i\underline{K}\cdot\underline{\rho} + i\Omega t}, \tag{140b}$$

$$E_1(x,y,z,t) = i \cdot \frac{\sigma A\Omega\, B_p[\cos\phi_D \cos\kappa + i\sin\phi_D]}{4K(1 + 3\cos^2\phi_D)^{1/2}}\, e^{-Ky - i\underline{K}\cdot\underline{\rho} + i\Omega t} \tag{140c}$$

These fields are resolved along the unit vectors $\underline{\ell}_1$, $\underline{\ell}_2$, $\underline{\ell}_3$ in Fig. 3. The E-mode field complex, Eqs. (139), and the H-mode field complex, Eq. (140), may be interpreted, individually, as a classic electromagnetic surface wave transporting power along the $\underline{K}$ direction (direction $\underline{\ell}_1$ in Fig. 3). The group velocity and the phase velocity of these electromagnetic surface waves are identical to the group and phase velocity of the hydrodynamic surface wave. It is also interesting to observe that

these electromagnetic surface waves are structurally indistinguishable from surface waves that would arise in air above a dielectric interface for plane waves incident from within a dielectric half space and totally reflected at the interface. The refractive index of such an equivalent dielectric would have to be extremely large. It is given by*

$$n = \frac{c}{v_p \sin\theta},$$

where v_p is the phase velocity of the hydrodynamic surface wave, θ the angle of incidence of the plane wave from within the dielectric and c the speed of light in vacuo. One then finds that $n \sin\theta \sim 10^7$. Referring to Fig. 3 and Eqs. (139) and (140), one observes that the amplitudes of the E-mode surface wave components are maximum when $\underline{K}$ is normal to the horizontal component of the earth's field and that they vanish for $\underline{K}$ aligned with the earth's magnetic field. One also notes that the vertical component of the earth's field does not contribute to the E-mode fields. On the other hand, the H-mode surface wave amplitudes depend both on the vertical and on the horizontal components of the earth's field. When the vertical component of the earth's field is zero (e.g., in the equatorial regions) the H-mode surface wave components vanish for $\underline{K}$ parallel to the earth's field, and are largest when the surface wave travels in the direction normal to the earth's field. The real power flow in each individual surface wave mode is directed along the propagation vector $\underline{K}$. The complex Poynting vector for each mode is

*The vertical attenuation of these electromagnetic surface waves is actually $K\sqrt{1-\left(\frac{v_p}{c}\right)^2}$, which factor has been approximated by K in (139) and (140) since $v_p/c \ll 1$.

$$\underline{P}_E = \frac{1}{2}\left[\underline{\ell}_1(E_2 H_3^*) - \underline{\ell}_2(E_1 H_3^*)\right]$$

$$= \left(\frac{\Omega^3 \varepsilon_o}{K}\right) \frac{A^2 B_D^2 \cos^2 \phi_D}{2(1 + 3\cos^2 \phi_D)} \sin^2 x\, e^{-2Ky} (\underline{\ell}_1 - i\underline{\ell}_2) \quad , \tag{141a}$$

$$\underline{P}_H = \frac{1}{2}\left[-\underline{\ell}_1(E_3 H_2^*) + \underline{\ell}_2(E_3 H_1^*)\right]$$

$$= \left(\frac{\Omega^3 \mu_o}{32K^3}\right) \frac{A^2 \sigma^2 B_D^2[\cos^2 \phi_D \cos^2 x + \sin^2 \phi_D]}{(1 + 3\cos^2 \phi_D)} e^{-2Ky} (\underline{\ell}_1 + i\underline{\ell}_2) \quad . \tag{141b}$$

The real part of $\underline{P}_E$ and $\underline{P}_H$ is directed along $K(\underline{\ell}_1)$ and represents real power transport by each surface wave; the imaginary parts are directed along $y(\underline{\ell}_2)$, and correspond to time-averaged stored energy required to support the traveling surface wave. The negative sign of the imaginary part of $\underline{P}_E$ indicates that the stored energy is predominantly in the electric field, while the opposite sign of the imaginary part of $\underline{P}_H$ shows that for the H-mode the time-averaged stored energy is predominantly magnetic. Note that the real and imaginary parts of $\underline{P}_{E,H}$ are equal in magnitude. This is a consequence of the equality of the transverse and longitudinal ($\underline{K}$-directed) field components of the eponymous modes. The complex Poynting vector $\underline{P}$ for the total surface wave complex comprising the E and H-surface wave mode is

$$\underline{P} = \underline{P}_E + \underline{P}_H + \underline{\ell}_3\, e^{-2Ky} \frac{A^2 \sigma B_D^2 \Omega^2 \cos \phi_D \sin x}{4K(1 + 3\cos^2 \phi_D)} \left[\sin \phi_D + i\cos \phi_D \sin \phi_D\right] . \tag{142}$$

The last term represents coupling between the two surface wave modes so that $\underline{P} \neq \underline{P}_E + \underline{P}_H$, in general. In particular, this coupling leads to real *power flow along* $\underline{\ell}_3$, (i.e., orthogonal

to the direction of propagation of each of the modes) of the amount

$$\operatorname{Re} \underline{P} \cdot \underline{\ell}_3 = e^{-2Ky} \frac{A^2 \sigma B_D^2 \Omega \cos \phi_D \sin \phi_D}{4K(1 + 3\cos^2 \phi_D)} \sin w \quad . \tag{143}$$

If $w = 0$ or (and) $B_{oy} = 0$, this term vanishes, in which case only an H-mode is excited. Similarly, there is no coupling whenever $B_{ox} = 0$. (Only the E-mode is excited.) The ratio of the magnitude of the real power transported by the E-mode to that transported by the H-mode is

$$\frac{\operatorname{Re} P_E}{\operatorname{Re} P_H} = \frac{16 K^2}{\sigma^2} \left(\frac{\varepsilon_o}{\mu_o} \right) \frac{\cos^2 \phi_D \sin^2 w}{(\cos^2 \phi_D \cos^2 w + \sin^2 \phi_D)} \quad . \tag{144}$$

Since $\sigma = 4$ mho/m, $\frac{\varepsilon_o}{\mu_o} \approx (377)^{-2}$ ohm^{-2}, and K for surface waves is on the order of unity or less, the power transported by the E-mode appears much smaller than that carried by the H mode except for $\phi_D \approx 0$ and $w \approx \pi/2$, i.e., when the latter vanishes. It turns out that when both mode contributions are non-vanishing, the dominant contributor to the real part of the Poynting vector is not the E-mode or H-mode taken in isolation, but the E to H-mode coupling term given by (143). Under these conditions the net real power flow is directed nearly normally to the propagation vector $\underline{K}$, i.e., along $\underline{\ell}_3$ in Fig. 3. This may be seen from the following considerations. The *magnitude* of the real part of the total Poynting vector in (142) may be written as follows:

$$\operatorname{Re} P = \frac{1}{32} e^{-2Ky} \frac{A^2 B_D^2 \sigma^2 \mu_o \Omega^3}{K^3(1 + 3\cos^2 \phi_D)} \cdot \left\{ \left[1 - \left(1 - \frac{16K^2\varepsilon_o}{\sigma^2\mu_o} \right) \cos^2 \phi_D \sin^2 w \right]^2 + \left[\frac{8K^2 \cos \phi_D \sin \phi_D \sin w}{\sigma \mu_o \Omega} \right]^2 \right\}^{\frac{1}{2}} . \tag{145}$$

If we denote the angle between the propagation vector $\underline{K}$ and the direction of Re $\underline{P}$ by θ_K, one finds

$$\tan \theta_K = \frac{\frac{4K^2}{\sigma\mu_o\Omega} \sin 2\phi_D \sin w}{1 - \left(1 - \frac{16K^2\varepsilon_o}{\sigma^2\mu_o}\right) \cos^2 \phi_D \cos^2 w} \quad . \tag{146}$$

The geometrical relationship between Re $\underline{P}$ and $\underline{K}$ is shown in Fig. 4, below:

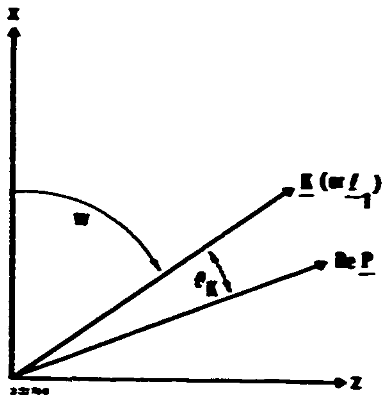

FIGURE 4.

Thus, if $w \neq 0$, only in the equatorial region ($\phi_D = 0$) and the polar region ($\phi_D = \pi/2$) is the real part of the Poynting vector directed exactly along $\underline{K}$. At intermediate latitudes, say $\Lambda = \pi/4$, (see Eq. (118)), $\tan \phi_D = 2$ and one obtains

$$\tan \theta_K \approx \frac{16K^2}{5\sigma\mu_o\Omega} \; \frac{\sin w}{1 - \frac{1}{5}\cos^2 w} \quad .$$

Taking $\Omega = 1$ rad sec $K = 1/9.8$ rad/meter $\tan \theta_K \approx 6.6 \times 10^3 \dfrac{\sin w}{1 - \frac{1}{5} \sin w}$.

Thus, unless w is nearly zero (i.e., the hydrodynamic surface wave is traveling almost exactly along the direction of the horizontal component of the geomagnetic field), $\theta_K \approx \frac{\pi}{2}$, viz., the direction of electromagnetic energy transport is nearly normal to the direction of propagation of the hydrodynamic surface wave.

To obtain some numerical estimates of the magnitude of the electromagnetic powers, i.e., Re P, consider first $\phi_D = 0$ (equatorial region). Then

$$\text{Re } P \approx \frac{1}{2} A^2 e^{-2Ky} \frac{\sigma^2 \mu_o \Omega^3}{16K^3} \cdot \frac{B_D^2}{4} \cos^2 w \quad .$$

For $\Omega = 1$, $K = 1/9.8$ rad/m, $w = 0$ and wave height of 1m, one obtains at the ocean surface a power density of approximately 6×10^{-13} watts/m^2. In intermediate latitudes $\Lambda = \pi/4$

$$\cos \phi_D = \frac{1}{\sqrt{5}} \quad , \qquad \sin \phi_D = \frac{2}{\sqrt{5}} \quad ,$$

and

$$\text{ReP} \cong \frac{1}{2} A^2 e^{-2Ky} \frac{\sigma^2 \mu_o \Omega^3}{16K^3} \cdot \frac{5B_D^2}{8} \cdot \left\{ \left(1 - \frac{1}{5} \sin^2 w\right)^2 + \left(\frac{16K^2}{5\sigma\mu_o\Omega} \sin w\right)^2 \right\}^{1/2} \quad .$$

Again, for the same parameters as in the preceding one has at $y = 0$

$$\text{ReP} \cong 1.44 \times 10^{-12} \left\{ \left(1 - \frac{1}{5} \sin^2 w\right)^2 + \left(6.62 \times 10^3 \sin w\right)^2 \right\}^{1/2} \quad .$$

With $w = \frac{\pi}{2}$, this yields ReP $\cong 1.17 \times 10^{-10}$ watts/m^2, or about 200 times larger than in the equatorial region. Note that this increase comes about solely from the cross-power term. In the polar region the cross-power term again vanishes, and the power density is then carried by the H-mode alone. One obtains in this case $\approx 24 \times 10^{-13}$ watts/m^2 or 4 times the maximum power (at $w = 0$) in the equatorial zone. Although these power levels appear remarkably low, they are well above the ambient (300°K) thermal noise level. For example, a power density of 10^{-10} watts/m^2 impinging on a sensor of effective area of 1 cm^2 gives

10^{-14} watts. If one supposes that this power is contained in a 1 Hz bandwidth, then the equivalent "noise temperature" is

$$T_{eq} = \frac{10^{-14}}{1.38 \times 10^{-23}} \approx 7.24 \times 10^{9} \; {}^{\circ}K \quad ,$$

where 1.38×10^{-23} Joule/°K is the Boltzmann constant.

At low frequencies it is customary to specify detector sensitivities directly in terms of field quantities instead of electromagnetic power densities. When one compares field strength then the relative significance of the E-mode and H-mode surface wave contributions is somewhat different. Returning to the fundamental set of field quantities, Eqs. (139) and (140), one finds that because of the presence of ε_o as a factor in (139b) that the E-mode gives rise to a small magnetic field, while, on the other hand, the H-mode produces a small electric field. This is consistent with the numerical values of the two wave impedances: $Z^{(E)} = \frac{K}{\Omega\varepsilon_o}$ is very large, while $Z^{(H)} = \frac{\Omega\mu_o}{K}$ is very small. Thus, for the E-mode at $y = 0$ with

$B_{oz} \cong 3.12 \times 10^{-5}$ Tesla (equatorial zone), $A = 1$ m, $\Omega = 1$ rad/sec, $x = \frac{\pi}{2}$, $\left|E_2\right| = \left|E_3\right| \approx 31.2$ μ volt/m, while the magnetic field is

$$\mu_o |H_3| \cong 4\pi \times 10^{-7} \cdot \frac{10^{-6} \times 3.12 \times 10^{-5}}{36\pi \times 9.8}$$

$\approx 3.53 \times 10^{-23}$ Tesla $= 3.53 \times 10^{-11}$ pT, which is well outside the sensitivity range of present day magnetometers. On the other hand 31.2 μ volts/m is certainly a measurable quantity. The power carried by the E-mode is also small, viz., $3.12 \times 10^{-6} \times 3.53 \times 10^{-23}/4\pi \times 10^{-7}) \approx 8.76 \times 10^{-}$[illegible] watts/m^2. For an effective sensor area of 1 cm^2 this corresponds to a level well below thermal noise at 300°K. For the same parameters for the H-mode one has

$$\mu_o|H_2| = \mu_o|H_1| \cong 9.8 \times 3.12 \times 10^{-5} \times 4\pi \times 10^{-7}$$

$$\cong 3.84 \times 10^{-10} \text{ Tesla} = 384 \text{ pT},$$

which is well within the sensitivity range of current magnetic sensors. However, the corresponding electric field is small, viz., $|E_2| \approx (9.8)^2\ 4\pi \times 10^{-7} \times 3.12 \times 10^{-5}$

$\approx 3.76 \times 10^{-9}$ volt/m $= 3.76 \times 10^{-3}$ μ volt/m. The power carried by the mode is approximately 5×10^{-13} watt/m^2. For an effective sensor area of 1 cm^2 this yields $T_{eq} \approx 3.62 \times 10^8$ °K for a bandwidth of 1 Hz. In summary, the H-mode comprises detectable magnetic fields, and a detectable power density and an essentially nondetectable electric field. The E-mode comprises detectable electric fields but a nondetectable magnetic field and power density. It should be noted however that this low power density corresponds to the E-mode itself, and does not include its interaction with the H-mode. This interaction power density is usually larger than the intrinsic H-mode power density.

We should now like to comment on the connection between "full wave solution" in (139) and (140) and the corresponding fields obtained under the quasi-static approximation Eqs. (131) and (136). Clearly, in the quasi-static case H_3 in (139b) and E_3 in (140c) are taken as zero, the remaining field components are, of course, identical. This may be verified by setting $\alpha = w \quad h_s = \frac{i\Omega}{K} A\, e^{i\Omega t}$ in (131) and (136). Thus, in the quasi-static approximation, the electric field is the vertical and longitudinal (directed along $\underline{\ell}_1$) field of the E-mode, the small magnetic field in the transverse direction ($\underline{\ell}_3$) having been neglected. The only characteristic feature that may permit one to infer from the quasi-static result that the fields in reality are part of a propagating guided wave, is the 90-deg

phase relationship between the vertical and the longitudinal electric field components. Even though the transverse magnetic field itself is nonmeasurable, it plays an indispensable role in the mechanism of electromagnetic power transport. Finite (measurable) power is obtained by virtue of the high wave impedance of the E-mode. Thus, since

$$E_2 = - Z^{(E)} H_3 \ , \text{ where } Z^{(E)} = \frac{K}{\Omega \varepsilon_o} \cong 377 \frac{c}{v_p} \text{ with } \frac{c}{v_p} \sim 10^7,$$

the electric field E_2 is measurable even though $|H_3|$ is very small. Similar remarks apply to the H-mode. In this case the quasi-static result yields only the magnetic fields, the accompanying electric field E_3 transverse to the propagation direction being negligibly small. However, its energetic interaction (in the sense of power transfer) is facilitated by the fact that the wave impedance for the H-mode is very small, viz., $Z^{(H)} = \frac{\Omega \mu_o}{K} \cong 377 \frac{c}{v_p}$, so that $H_2 = E_3/Z^{(H)}$ is again a measurable quantity. Of course, even a single hydrodynamic surface wave generally generates both modes, so that transfer of electromagnetic power is not directed along $\underline{K}$. It is of interest to contrast this picture with that corresponding to a unidirectional internal wave. (For simplicity assume that it comprises only a single mode.) We first recall that the quasi-static approximation in this case yields only magnetic fields, which are quite similar in structure to the magnetic fields generated by a surface wave, and which we found to correspond to an H-mode. Evidently then, the internal wave generates no E-modes; consequently, the magnetic field components induced by an internal wave must correspond to an H-mode electromagnetic surface wave. Since the two magnetic field components have already been obtained under the quasi-static approximation, the only additional component needed to complete the characterization of this mode is the transverse electric field E_3. Clearly, this

component may be computed by multiplying the vertical component of the H field obtained under the quasi-static approximation by the wave impedance $Z^{(H)} = \frac{\Omega\mu_o}{K}$, where Ω is now related to K by the dispersion relationship for the internal wave. If more than one internal wave mode contributes, then the electric field E_3 is computed by summing the individual contributions of all modes. However, to the extent that the internal wave is unidirectional, the total induced electromagnetic surface wave still corresponds to a pure H-mode field and hence electromagnetic energy is necessarily transported only along the $\underline{K}$ direction.

VI. SPECTRA OF ELECTROMAGNETIC FIELDS INDUCED BY INTERNAL WAVES AND SURFACE WAVES

We shall now suppose that the expressions for the electromagnetic fields derived in the preceding section apply to a typical realization of a stochastic internal wave or surface wave process. These stochastic processes will be assumed wide sense stationary in time and spatially homogeneous in the transverse (x,z) plane. With the aid of the analytical expressions for internal wave and surface wave spectra presented in Appendix E and A, respectively, we shall derive the corresponding spectra for the induced electromagnetic field components and magnetic field gradients. These formulas will then be applied to compute internal-wave-induced magnetic field spectra above the ocean for the case of exponential stratification. For surface waves, the electric and magnetic field spectra will be obtained for the Pierson-Neumann surface wave spectrum model. We first consider internal waves.

A. SPECTRA OF COMPONENTS OF THE MAGNETIC FIELD ABOVE THE OCEAN SURFACE INDUCED BY INTERNAL WAVES

With the induced magnetic field resolved along the unit vectors $\underline{\ell}_1$, $\underline{\ell}_2$, $\underline{\ell}_3$ in Fig. 3, we define correlation functions for field components at a fixed height y above the ocean surface by

$$R_{\nu\mu}(\underline{\rho}',\underline{\rho}'',t',t'';y) = \langle B_\nu(\underline{\rho}',t';y)\, B^*_\mu(\underline{\rho}'',t'';y)\rangle \quad , \qquad (146)$$

with $\nu,\mu = 1,2,3$.

We now employ (109) together with (124) in (146) to obtain

$$R_{\nu\mu}(\underline{\rho}',\underline{\rho}'',t',t'';y) = \frac{(\sigma\mu_o B_p)^2}{(1+3\cos^2\phi_D)}$$

$$\cdot \iint_{-\infty}^{\infty} d\underline{K}' \iint_{-\infty}^{\infty} d\underline{K}'' \, e^{-i\underline{K}'\cdot\underline{\rho}'+i\underline{K}''\cdot\underline{\rho}''} \, e^{-(K'+K'')y} \, g_\nu(w',\alpha) g_\mu^*(w'',\alpha) \langle h(\underline{K}',t') h^*(\underline{K}'',t'')\rangle, \tag{147}$$

where

$$g_1(w,\alpha) = \frac{1}{2}\cos(w-\alpha)\,[\cos\phi_D \cos w - i\sin\phi_D] \quad , \tag{148a}$$

$$g_2(w,\alpha) = -\frac{i}{2}\,[\cos\phi_D \cos w - i\sin\phi_D] \quad , \tag{148b}$$

$$g_3(w,\alpha) = \frac{1}{2}\sin(w-\alpha)\,[\cos\phi_D \cos w - i\sin\phi_D] \quad . \tag{148c}$$

Using Eq. (123), and Eq. (E-56') in Appendix E, one finds that the statistical average in the integrand of (147) can be expressed as follows:

$$\langle h(\underline{K}',t')\, h^*(\underline{K}'',t'')\rangle$$

$$= \frac{1}{2}\sum_n L_n^2(K')\left[\psi_n(\underline{K}')e^{i\Omega_n(K')(t'-t'')} + \psi_n(-\underline{K}')e^{-i\Omega_n(K')(t'-t'')}\right]\delta(\underline{K}'-\underline{K}'') \quad , \tag{149}$$

where

$$L_n(K) = \int_{-\infty}^{0} \phi_n(y')e^{Ky'}\,dy' \quad . \tag{150}$$

The functions $\psi_n(\underline{K}')$ are real and nonnegative, and are proportional to the spatial spectra of the internal wave modes. Substituting (149) in (147) and setting $t'-t'' = \tau$, $\underline{\rho}'-\underline{\rho}'' = \underline{\rho}$ one obtains

$$R_{\nu\mu}(\underline{\rho},\tau,y) = \frac{(\sigma\mu_o B_p)^2}{2(1+3\cos^2\phi_D)} \cdot$$

$$\iint_{-\infty}^{\infty} d^2\underline{K} e^{-i\underline{K}\cdot\underline{\rho}} \epsilon^{-2Ky} g_{\nu\mu}(w,\alpha) \cdot \sum_n L_n^2(K)\left[\psi_n(\underline{K})e^{i\Omega_n(K)\tau} + \psi_n(-\underline{K})e^{-i\Omega_n(K)\tau}\right] , \quad (151)$$

where we have defined

$$g_{\nu\mu} = g_\nu g_\mu^* \quad . \qquad (152)$$

The elements of the temporal cross-spectral matrix $\Phi_{\nu\mu}(\underline{\rho},\omega,y)$ are given by the Fourier transform of (151) with respect to τ. Changing the variables of integration from the cartesian to the polar form and taking the Fourier transform one has, for $\omega \geq 0$,

$$\Phi_{\nu\mu}(\underline{\rho},\omega,y) = \pi\frac{(\sigma\mu_o B_p)^2}{(1+3\cos^2\phi_D)}$$

$$\sum_n e^{-2K_n(\omega)y} \frac{K_n(\omega)\, L_n^2[K_n(\omega)]}{v_{gn}(\omega)} \int_0^{2\pi} dw\, e^{-iK_n(\omega)\,\rho\cos(w-\theta)} g_{\nu\mu}(w,\alpha)\psi_n[K_n(\omega),w] ,$$

(153)

where $K_n(\omega)$ is the solution of $\Omega_n(K) = \omega$ for K, and $v_{gn}(\omega) = \left.\frac{d\Omega_n(K)}{dK}\right|_{K = K_n(\omega)}$ is the group speed of the nth internal wave mode. In the special case of $\rho = 0$ (the sensors are collocated), one obtains the spectral density matrix proper:

$$\Phi_{\nu\mu}(0,\omega,y) = \pi\frac{(\sigma\mu_o B_p)^2}{(1+3\cos^2\phi_D)}$$

$$\sum_n e^{-2K_n(\omega)y} \frac{K_n(\omega)\, L_n^2[K_n(\omega)]}{v_{gn}(\omega)} \int_0^{2\pi} g_{\nu\mu}(w,\alpha)\psi_n[K_n(\omega),w]\, dw \quad . \qquad (154)$$

At this stage no special assumptions have been made with regard to the excitation functions $\psi_n(K,w)$. If we assume that they are isotropic, i.e., that each ψ_n is independent of w, then a reference to the defining relations for $g_{\nu\mu}$, Eqs. (152) and (148) shows that

$$\Phi_{12}(0,\omega,y) = \Phi_{23}(0,\omega,y) = 0 \quad ,$$

while

$$\Phi_{13}(0,\omega,y) = \gamma_{13}[\Phi_{11}(0,\omega,y)\ \Phi_{33}(0,\omega,y)]^{-1/2} \quad ,$$

where

$$\gamma_{13} = -\frac{1}{2}\sin 2\alpha \ \frac{\cos^2\phi_D}{\sin^2\phi_D + \frac{1}{8}\cos^2\phi_D\ (1 + 2\sin^2\alpha)} \ .$$

In other words, vertical and horizontal components of the induced magnetic field are completely decorrelated. On the other hand, the two horizontal components are partially correlated with the correlation coefficient γ_{13} which depends only on α and ϕ_D. At the equator ($\phi_D = 0$), $|\gamma_{13}|$ reaches a maximum value of $\sqrt{3}/3$ at $\alpha = 30$ deg. Generally, we can take the spectral coherence function $\gamma_{\nu\mu}(0,\omega,y)$,

$$\gamma^2_{\nu\mu}(0,\omega,y) = \frac{|\Phi_{\nu\mu}(0,\omega,y)|^2}{\Phi_{\nu\nu}(0,\omega,y)\ \Phi_{\mu\mu}(0,\omega,y)} \leq 1 \quad , \qquad (155)$$

for $\nu \neq \mu$ as an indicator of the directionality of the internal wave spectrum. Thus, if the internal wave spectrum is perfectly directional, we find that the coherence function between any two orthogonal components equals unity for all α. The direction of propagation of such a stochastic wave train can, in principle, be determined by a spectral correlation measurement of γ_{12} or γ_{23}. Consider, for example, a magnetic field sensor that provides

a simultaneous measurement of a horizontal and the vertical magnetic field component. If this sensor is rotated about a vertical axis, the measured horizontal component coherence function will undergo several excursions between zero and unity. In particular, if the direction of propagation of the wave train is defined by $w = w_o$, we find that $\gamma_{12} = 1$ whenever $w_o = \alpha \pmod{\pi}$ while $\gamma_{23} = 1$ for $w_o = \alpha \pm \pi/2 \pmod{\pi}$. Clearly, if the internal wave field is only partly directional, then results of such a spectral coherence measurement can be used to estimate the degree of anisotropy of the internal wave spectrum. Viewed from a slightly different perspective, the discrimination on the basis of directionality in wave number space arises from the angular dependence of the $g_{\nu\mu}(w,\alpha)$. These wave number projection factors provide enhancement of cross-spectral power of a unidirectional internal wave field relative to an isotropic one. A quantitative measure of this enhancement is the directive gain $G_{\nu\mu}$, defined by

$$G_{\nu\mu}(w_o;\alpha) = \frac{g_{\nu\mu}(w_o,\alpha)}{\frac{1}{2\pi}\int_0^{2\pi} g_{\nu\mu}(w,\alpha)dw} \quad . \tag{156}$$

When $\nu = \mu$, the maximum of $G_{\nu\mu}$ may be interpreted as the "maximum power gain" relative to an isotropic internal wave background when an ideal magnetic field component detector is used to measure component ν. The gain is rather modest. As may be seen from an examination of Eq. (148), the largest directive discrimination obtains for one of the two horizontal components. We find that for $\alpha = \pi/2$, G_{11} attains a maximum value of 2 for $\phi_D = 0$ (equatorial zone) with $w_o = \pi/4 \pmod{\pi/2}$; for $\alpha = 0$ the maximum directive gain for this component at $\phi_D = 0$ equals 8/3 with $w_o = 0$. Thus, the intrinsic spatial wave number filtering properties of a magnetic field component sensor afford only marginal discrimination between unidirectional and isotropic internal wave spectra.

When more than one collocated component sensor is employed, each responding to a different orthogonal component of the induced field, additional discrimination is possible on the basis of spectral correlation. Note that in Eq. (156) $G_{2\mu}$ is infinite whenever $\mu \neq 2$, indicating potentially perfect discrimination between isotropy and unidirectionality. This is just a restatement of the result obtained earlier in terms of the spectral coherence function. In a practical situation the discrimination would, of course, not be perfect. Nevertheless, multiple component sensors would generally afford a greater degree of discrimination than a single component sensor.

In order to obtain numerical estimates of the magnetic field spectra one must have information on the partitioning of energy among modes in mode-wave number space. We shall employ the hypothesis of Milder [9] according to which the modal constituents comprising the wave number energy spectrum of internal waves are distributed in proportion to the square of their phase velocities. The functions $\psi_n(\underline{K})$ are then given by Eq. (E-73) of Appendix E. The correlation functions in Eq. (151) then become

$$R_{\nu\mu}(\underline{\rho},\tau,y) = \frac{1}{2\pi} \frac{(\sigma\mu_o B_p)^2}{1 + 3\cos^2\phi_D}$$

$$\iint_{-\infty}^{\infty} d^2\underline{K}\, e^{-i\underline{K}\cdot\underline{\rho}}\, e^{-2Ky} \frac{g_{\nu\mu}(\kappa,\alpha)}{K^3} \sum_n L_n^2(K)\, \Omega_n^4(K) \left[I(\underline{K})e^{i\Omega_n(K)\tau} + I(-\underline{K})e^{-i\Omega_n(K)\tau}\right]. \tag{157}$$

The quantity $I(\underline{K})$ is an excitation function which depends on $\underline{K}$ but not on n. It is this last feature and not the functional form of $I(\underline{K})$ which is crucial to the validity of the closed form expressions for the spatial spectra given in the sequel and Appendix E. The spatial cross-spectrum is evidently given by

$S_{\nu\mu}(\underline{K},y) =$

$$\frac{1}{2\pi}\frac{(\sigma\mu_o B_p)^2}{1+3\cos^2\phi_D} \cdot e^{-2Ky}\frac{g_{\nu\mu}(w)}{K^3}[I(\underline{K})+I(-\underline{K})]\sum_n L_n^2(K)\,\Omega_n^4(K) \quad , \tag{158}$$

so that

$$R_{\nu\mu}(\underline{\rho},0,y) = \iint_{-\infty}^{\infty} e^{-i\underline{K}\cdot\underline{\rho}}\, S_{\nu\mu}(\underline{K},y)\, d^2\underline{K} \quad . \tag{159}$$

The sum in (158) can be expressed explicitly in terms of the Väisälä frequency. The required formulas are given in Eqs. (F-6) (F-7) and (F-8) of Appendix F. If in addition we employ the definition of $L_n(K)$ in Eq. (150), we obtain

$$\sum_n L_n^2(K)\,\Omega_n^4(K) = K^4\int_{-\infty}^{0} N^2(y'')\left[\int_{-\infty}^{0} g(y'',y')e^{Ky'}\,dy'\right]^2 dy'' \quad , \tag{160}$$

where $g(y'',y')$ is given by* (F-8) since here we are assuming a deep ocean. After carrying out the integration one finds

$$\sum_n L_n^2(K)\,\Omega_n^4(K) = \frac{K^2}{4}\int_{-\infty}^{0} y^2N^2(y)\, e^{2Ky}\, dy \quad . \tag{161}$$

The right side can be evaluated for any specified Väisälä frequency profile**. Therefore, the effect of different oceanic

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*Note that $g(y'',y') = g(y',y'')$.

**It is important to note that in Eq. (161) the Väisälä frequency profile $N(y)$ must tend to zero as $y \to -\infty$, since we used the $g(y'',y')$ function for the deep ocean, Eq. (F-8). Thus Eq. (161) is *not* valid for a constant N. A formula similar to Eq. (161) can be derived by using Eq. (F-7), which formula would then hold for an ocean of finite depth and arbitrary $N(y)$(in particular, for $N(y)$ = constant).

stratifications on the spatial cross-spectrum of the induced components of the magnetic field can be computed *without the knowledge* of the internal wave eigenfunctions and dispersion relations.

The integral in Eq. (161) will be recognized as the Laplace transform of $y^2N^2(y)$. From the asymptotic theory of Laplace transforms we know that for a continuous profile the behavior of this integral for large K is determined by $y^2N^2(y)$ and its derivatives for small values of y, i.e., near the ocean surface. Thus, a jump in $y^2N^2(y)$ at $y = 0$ gives the asymptotic decay of $1/K$, while a jump in the first derivative constrains the asymptotic decay to $1/K^2$.

To completely characterize the behavior of the spatial spectrum in (158) one must specify the excitation function $I(\underline{K})$. We shall assume an excitation function that is isotropic in wave number space with a dependence on the wave number of the form $I(K) = CK^{-p}$; the constants C and p are usually of semi-empirical origin. (See discussion in Appendix E, Section D.) Reasonable values of p appear to be between two and unity. Employing this excitation function in (158) together with (161), gives the following result for the spatial cross-spectrum:

$$S_{\nu\mu}(\underline{K},y) = \frac{C}{4\pi}\,\frac{(\sigma\mu_0 B_p)^2}{1+3\cos^2\phi_D}\,g_{\nu\mu}(w)\,e^{-2Ky}K^{-p-1}\int_{-\infty}^{0} y^2N^2(y)\,e^{2Ky}\,dy\,. \qquad (162)$$

Unlike the assumed internal wave excitation function (and, necessarily, also the spatial spectrum of fluid particle displacement), the magnetic field component cross-spectra are not isotropic but depend on w through the trigonometric terms entering in $g_{\nu\mu}(w)$. At any point above the ocean surface, the asymptotic decay of $S_{\nu\mu}$ for large K is dominated by the exponential factor.

We shall find in Chapter VII that the spatial spectrum can be used to predict the approximate behavior of the temporal spectrum observed from moving platforms. Here we shall use it only to compute the r.m.s. induced magnetic field. Clearly, from the definition of the correlation function, the average of the square of any orthogonal component is

$$\langle B_{\nu\nu}^2(\underline{\rho},t,y)\rangle = R_{\nu\nu}(0,0,y) = \iint_{-\infty}^{\infty} d^2\underline{K}\, S_{\nu\nu}(\underline{K},y) = \int_0^{\infty} K dK \int_0^{2\pi} dw\, S_{\nu\nu}(K,w,y) \ . \quad (163)$$

The total r.m.s. induced magnetic field at any point above the ocean surface is

$$B_{rms}(y) = \sqrt{\sum_{\nu=1}^{3} \langle B_{\nu\nu}^2\rangle} \ . \quad (164)$$

The integration over w involves the wave number projection factors (148) and (152). One finds

$$\int_0^{2\pi} g_{11}(w,\alpha)dw = \frac{\pi}{2}\,[\sin^2\phi_D + \frac{1}{8}\cos^2\phi_D(1+2\cos^2\alpha)] \ , \quad (165a)$$

$$\int_0^{2\pi} g_{22}(w,\alpha)dw = \frac{\pi}{2}\,[\sin^2\phi_D + \frac{1}{2}\cos^2\phi_D] \ , \quad (165b)$$

$$\int_0^{2\pi} g_{33}(w,\alpha)dw = \frac{\pi}{2}\,[\sin^2\phi_D + \frac{1}{8}\cos^2\phi_D(1+2\sin^2\alpha)] \ . \quad (165c)$$

Employing Eq. (162) in Eq. (163) and adding the three integrals gives

$$B_{rms}^2(y) = \frac{C}{8}(\sigma\mu_o B_p)^2 \frac{1+2\sin^2\phi_D}{(1+3\cos^2\phi_D)} \cdot \int_0^{\infty} dK\, e^{-2Ky} K^{-p} \int_{-\infty}^{0} y'^2 N^2(y') e^{2Ky'} dy' \ . \quad (166)$$

As expected, the total induced r.m.s. field is independent of the horizontal orientation of the geomagnetic field, but depends on the magnetic dip angle ϕ_D. There is a difficulty in (166) with the integration over K, in that for typical profiles and wave number decay constants, p, the singularity at K = 0 will cause the integral to diverge. This is simply a consequence of the assumption that the excitation function I(K) maintains the power law behavior down to K = 0. In truth, the whole theory, both in its electromagnetic and the hydrodynamic aspects, cannot reasonably be expected to apply to arbitrarily long wavelengths. Thus, in the internal wave part, we have introduced simplifications to exclude the long wavelength inertial range, while in the electromagnetics part we have relied on the quasi-static approximation with its intrinsic limit on the maximum permissible length scale. The simplest way to eliminate the divergence problem in (166) is to truncate the lower limit of integration to some non-zero value $K = K_c$. This is also done in the theoretical discussion of internal waves in Appendix E. We now assume an exponentially decreasing Väisälä frequency profile of the form $N(y) = N(0) \exp y/b$. The inner integral in (166) then yields

$$\int_{-\infty}^{0} y^2 N^2(y) e^{2Ky}\, dy = \frac{N^2(0) b^3}{4 \quad (Kb+1)^3} , \tag{167}$$

and the formula for the r.m.s. field takes on the special form

$$B^2_{rms}(y) = \frac{Cb^3 N^2(0)}{32} (\sigma\mu_o B_p)^2 \frac{1+2\sin^2\phi_D}{(1+3\cos^2\phi_D)} \cdot \int_{K_c}^{\infty} dK e^{-2Ky} K^{-p} (Kb+1)^{-3} . \tag{168}$$

For p = 2 the constant C is given by Eq. (E-125). Substituting this in (168) and evaluating the integral for p = 2 and y = 0, yields

$$B^2_{rms}(0) =$$

$$(\sigma\mu_o B_F)^2 \frac{1+2\sin^2\phi_D}{(1+3\cos^2\phi_D)} \cdot \frac{Eb}{32\pi\rho_o} \cdot \frac{\nu_c}{1-\nu_c \ln \frac{1+\nu_c}{\nu_c}} \left[\frac{3\nu_c+\nu_c^{-1}+4.5}{(1+\nu_c)^2} - 3 \ln \frac{1+\nu_c}{\nu_c} \right], \quad (169)$$

where $\nu_c = K_c b$, E is the average internal wave energy density integrated over the vertical water column, and ρ_o is average water density. Using the parameters extrapolated from Ref. [8] as discussed in Appendix E, we obtain $\nu_c = .327$, $E = .382 \times 10^4$ joules/m², and b = 1300 m. With these numerical constants the total r.m.s. field is

$$B_{rms}(0) \approx \sqrt{\frac{1+2\sin^2\phi_D}{1+3\cos^2\phi_D}} \times 13.76 \times 10^{-10} \text{ Tesla} . \quad (170)$$

The functional form of the decay of this field with increasing y can be determined by carrying out the integration in (168). Evidently, the decay is not purely exponential, as it is for each individual spectral component.

To measure the magnitude of the r.m.s. magnetic field as given by (170) requires a total field sensor that responds equally to all spectral components in the temporal frequency domain. The relative contribution of the spectral constituents to the r.m.s. field is determined by the temporal spectrum of the total field B(t). This spectrum is defined by the relationship

$$\langle B(t+\tau,y)\, B(t,y)\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{BB}(\omega,y) e^{i\omega\tau}\, d\omega \quad . \quad (171)$$

In the special case $\tau = 0$ one obtains

$$\langle B^2(t,y)\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{BB}(\omega,y)\, d\omega \quad . \quad (172)$$

The r.m.s. field can also be computed in the following alternative fashion:

$$\langle B^2(t,y)\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\Phi_{11}(\omega,y) + \Phi_{22}(\omega,y) + \Phi_{33}(\omega,y)\right] d\omega \quad . \tag{173}$$

The three spectra in the integrand are given by (154) and are just the temporal spectra of the individual components. It is important to note that $\Phi_{BB} \neq \Phi_{11} + \Phi_{22} + \Phi_{33}$, as one might be tempted to conclude by equating the integrands in (172) and (173). The mere fact that the integration in each of the two cases yields identical results gives one no information on the relationship between the two integrands. Clearly, there are many different functions ("spectra") from which the mean of $B^2(t,y)$ may be computed through integration. For example, the spatial spectrum in (163) is also such a function.

The determination of the spectrum of the total field requires the knowledge of the joint probability density function of the magnetic field components, since one must be able to compute the average

$$\langle B(t+\tau)\, B(t)\rangle = \langle \sqrt{B_1^2(t+\tau) + B_2^2(t+\tau) + B_3^2(t+\tau)}\ \sqrt{B_1^2(t) + B_2^2(t) + B_3^2(t)} \rangle \quad . \tag{174}$$

This operation can be carried out, for example, when the induced magnetic field components are assumed to obey joint Gaussian statistics. The algebraic manipulations are rather involved and we shall not carry them out. However, it should be apparent even without a detailed calculation that as a result of the complete overlap in frequency of the three component spectra Φ_{11}, Φ_{22}, Φ_{33} the spectrum of the total magnetic field will occupy a much larger bandwidth than the spectra of the individual components*.

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*The mechanism generating these additional frequency components is, of course, the same as in the run of the mill envelope detector when used without a low pass filter.

In the following we shall deal only with the temporal spectra of the individual components. We use the mode partitioning hypothesis as in (157) and again specialize the excitation function to its isotropic form and power law dependence on the wave number. The integrations over the $g_{\nu\mu}(\omega,\alpha)$ are then carried out as in (165) so that the temporal spectra for the three components become

$$\Phi_{11}(\omega,y) = \frac{\sin^2\phi_D + \frac{1}{8}\cos^2\phi_D(1+2\cos^2\alpha)}{4[1+3\cos^2\phi_D]}\, S_c(\omega,y)\ , \qquad (175a)$$

$$\Phi_{22}(\omega,y) = \frac{\sin^2\phi_D + \frac{1}{2}\cos^2\phi_D}{4[1+3\cos^2\phi_D]}\, S_c(\omega,y)\ , \qquad (175b)$$

$$\Phi_{33}(\omega,y) = \frac{\sin^2\phi_D + \frac{1}{8}\cos^2\phi_D(1+2\sin^2\alpha)}{4[1+3\cos^2\phi_D]}\, S_c(\omega,y)\ . \qquad (175c)$$

The function $S_c(\omega,y)$ will be referred to as the normalized component spectrum. It is given by

$$S_c(\omega,y) = 2\pi\ C(\sigma\mu_c B_p)^2\omega^4 \sum_n e^{-2K_n(\omega)y}\, \frac{[K_n(\omega)]^{-p-1}\ L_n[K_n^2(\omega)]}{K_n(\omega)\ v_{gn}(\omega)}\ . \qquad (176)$$

The ω^4 factor in this expression arises from the identity $\omega \equiv \Omega_n[K_n(\omega)]$.

In order to compute $S_c(\omega,y)$ for a specified Väisälä frequency profile one must determine the explicit form of the eigenfunctions and dispersion relations. Recall that such detailed information is not needed in the computation of the spatial spectra, Eq. (162), which are determinable directly from the Väisälä frequency profile without the knowledge of the

eigenfunctions. This simplified state of affairs arises entirely from our ability to carry out the sum of Eq. (160). In the temporal spectrum, Eq. (176), an analogous summation does not prove possible since the modal index, n, also enters in the factors $[K_n(\omega)]^{-p-1}/K_n(\omega)\ v_{gn}(\omega)$. These provide additional amplitude weighting that modifies the relative distribution of energy in frequency space.

We now proceed to apply formula Eq. (176) to an exponentially stratified ocean. The eigenfunctions for this case are given by Eq. (E-107). The spectra will be expressed in terms of the normalized angular frequency

$$\eta = \frac{\omega}{N(0)} \quad ,$$

N(0) being the maximum Väisälä frequency. We also define the dimensionless variable ν, $\nu = Kb$, and rewrite the dispersion relationship in Eq. (E-104) in the following normalized form:

$$\eta = \frac{\nu}{\chi_{n;\nu}} \quad , \tag{177}$$

where $\chi_{n;\nu}$ is nth root of the νth order Bessel function. For each n, Eq. (177) has one real solution for ν, which we denote by ν_n. If one also employs the formula for the group speed given in Eq. (E-110), the normalized component spectrum in Eq. (176) may be shown to reduce to

$$S_c(y,\omega) = 4\pi(\sigma\mu_c B_p)^2\, C\, b^{p+2}\, N(0) \sum_{n=1}^{\infty} s_n^{(p)}(\eta,y) \quad , \tag{178}$$

where the sum is over the dimensionless spectra

$$s_n^{(p)}(\eta,y) = \frac{e^{-2\nu_n \frac{y}{b}}\, \eta^{2\nu_n+3}\, \nu_n^{-2\nu_n-p-1} \left[\int_0^{\nu_n/\eta} J_{\nu_n}(t)\, t^{\nu_n-1}\, dt\right]^2}{J^2_{\nu_{n+1}}(\nu_n/\eta) - 2\eta^2 \int_0^{\nu_n/\eta} J^2_{\nu_n}(t)\, \frac{dt}{t}} \quad . \tag{179}$$

Note that the dimensionless quantity ν_n is a function of η, since $\nu_n(\eta) \equiv K_n(\omega)b$. The constant C is a function of p. For p = 1 and p = 2 it is given in Eqs. (E-124) and (E-125), respectively. With b = 1300 m, the constant multiplier appearing in front of Eq. (178) becomes

$$4\pi(\sigma\mu_o B_p)^2 C b^{p+2} N(0) = \begin{cases} 2.25 \times 10^{11} \ (\mathrm{pT})^2/\mathrm{Hz} \ ; \ p = 2 \ , \\ 2.66 \times 10^{11} \ (\mathrm{pT})^2/\mathrm{Hz} \ ; \ p = 1 \ . \end{cases}$$

For these two cases Eq. (178) was evaluated numerically. The results are shown in Figs. 5 and 6. The frequency is normalized to the maximum Väisälä frequency of .833 x 10^{-3}Hz. Figure 5 shows the spectrum of the horizontal component, viz., Eq. (175a). Figure 6 gives the plot of the normalized component spectrum $S_c(\omega,y)$ from which the spectra in Eqs. (175a,b,c) can be obtained for any geographical location and relative orientation of the magnetic field (see Fig. 3, p. 60). The largest difference in levels between case p = 1 and case p = 2 is at the ocean surface, and as one approaches the Väisälä frequency.

B. SPECTRA OF MAGNETIC FIELD GRADIENTS ABOVE THE OCEAN SURFACE INDUCED BY INTERNAL WAVES

The expressions for the spectra of magnetic gradients can be obtained by a slight modification of the expressions for the field components. We shall be interested only in the three orthogonal gradients, G_{12}, G_{23}, and G_{13}, whose Fourier transforms are related to the transforms of the field components by Eq. (125). The correlation function between $G_{\nu\mu}$, G_{rs}, when measured at the same height above the ocean, will be denoted by the four index quantity $R_{\nu\mu;rs}(\underline{\rho},\tau,y)$. Its general form follows from the correlation function for field components, Eq. (151), by including the additional factor K^2 and taking account of the slight modification in the angularly dependent factors $g_{\nu\mu}$, as determined from the inspection of Eq. (125). We now write these new wave

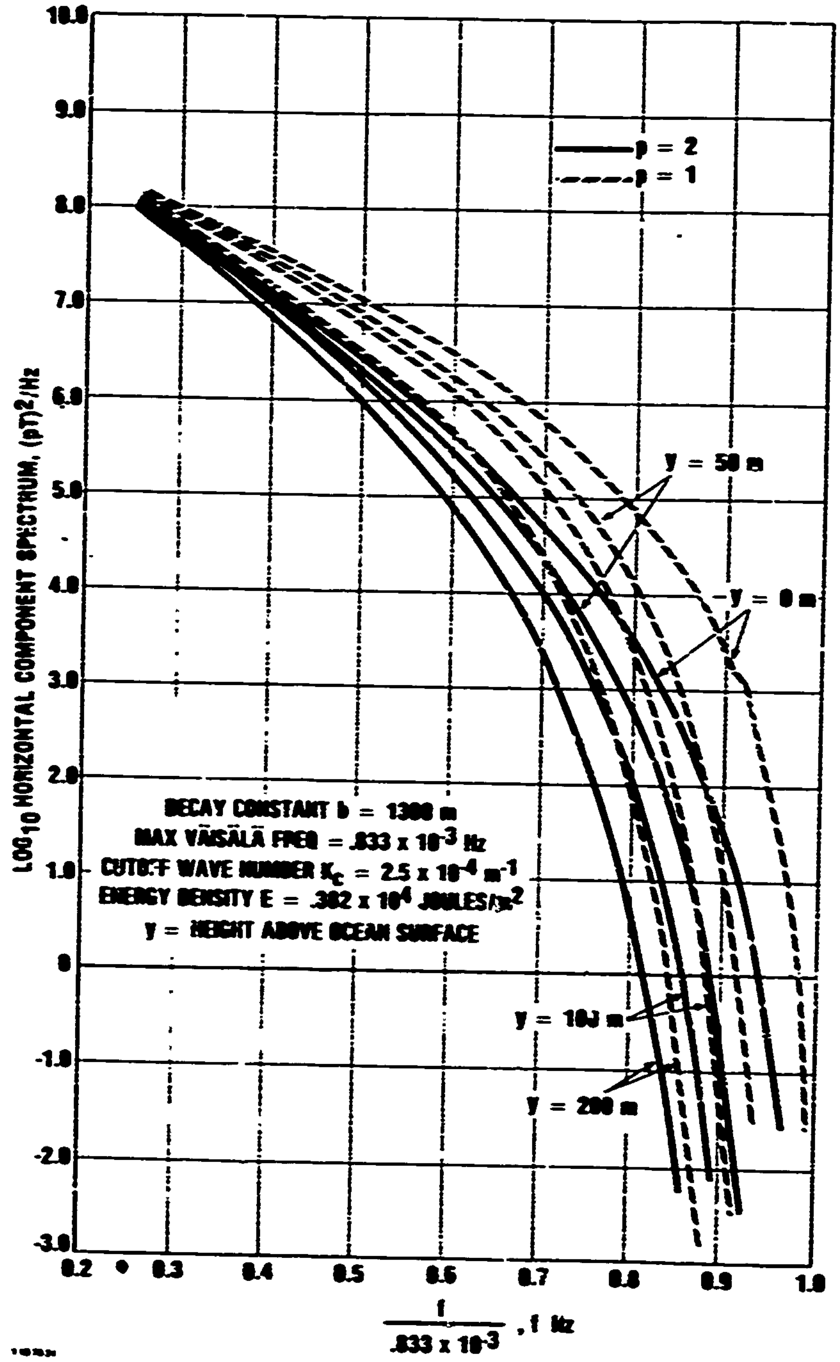

FIGURE 5. Horizontal component of magnetic field induced by internal waves in an exponentially stratified ocean ($\alpha = 0$, $\phi_D = 0$). The small departure from the monotonic behavior of the curves for $y = 0$ in Figs. 5-8 is *not* an artifact of the graphical representation of the data but can actually be explained in terms of the decay characteristics of higher order internal wave modes.

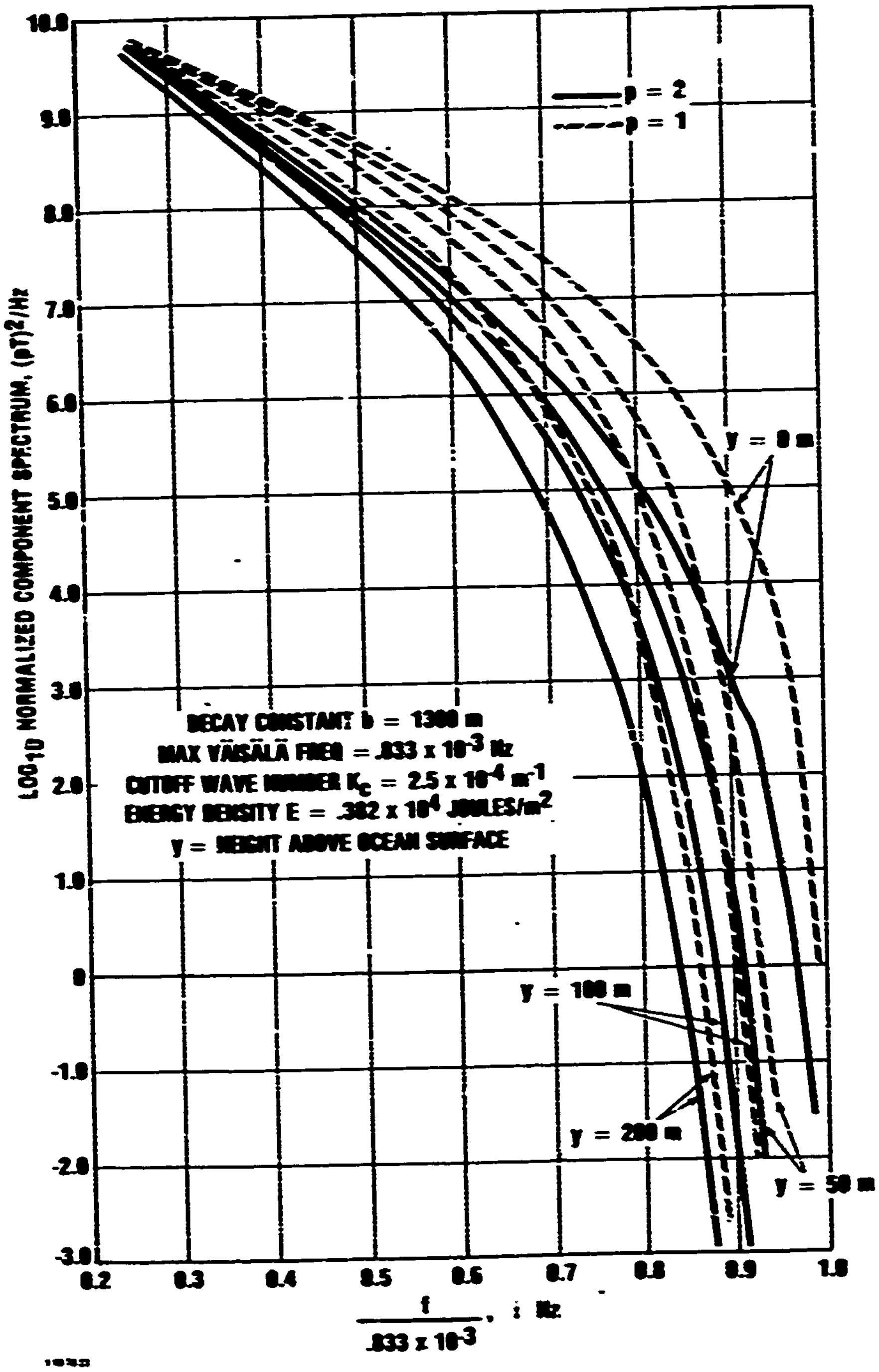

FIGURE 6. Magnetic field above the ocean surface induced by internal waves in an exponentially stratified ocean

number projection factors as quadruple index quantities $g_{\nu\mu;rs}(w,\alpha)$. The correlation function for magnetic field gradients is then

$$R_{\nu\mu;rs}(\underline{\rho},\tau,y) = \frac{(\sigma\mu_o \bar{B}_p)^2}{1 + 3\cos^2\phi_D}$$

$$\iint_{-\infty}^{\infty} d^2\underline{K}\, K^2\, e^{-i\underline{K}\cdot\underline{\rho}}\, e^{-2Ky}\, g_{\nu\mu;rs}(w,\alpha) \sum_n L_n^2(K) \left[\psi_n(\underline{K})\, e^{i\Omega_n(K)\tau} + \psi_n(-\underline{K})\, e^{-i\Omega_n(K)\tau} \right] . \tag{180}$$

The additional factor of 2 will be accounted for in the definition of $g_{\nu\mu;rs}$. With the aid of Eq. (132) we have, for any set of directions $\underline{\ell}_\nu$, $\underline{\ell}_\mu$, $\underline{\ell}_r$, $\underline{\ell}_s$,

$$g_{\nu\mu;rs}(w,\alpha) = (\underline{\ell}_\nu \cdot \underline{a})(\underline{\ell}_\mu \cdot \underline{a})(\underline{\ell}_r \cdot \underline{a}^*)(\underline{\ell}_s \cdot \underline{a}^*) \frac{|\underline{a}\cdot\underline{B}_o|^2}{|\underline{B}_o|^2} . \tag{181}$$

For the three orthogonal gradients of interest these wave number projection factors are obtained with the aid of Eq. (120):

$$g_{12;12}(w,\alpha) = \frac{1}{8}\cos^2(w-\alpha)\left[\cos^2\phi_D \cos^2 w + \sin^2\phi_D\right] = \frac{1}{2}\, g_{11}(w,\alpha) \quad , \tag{182a}$$

$$g_{12;23}(w,\alpha) = \frac{1}{16}\sin 2(w-\alpha) \left[\cos^2\phi_D \cos^2 w + \sin^2\phi_D\right] = \frac{1}{21}\sin(w-\alpha) g_{12}(w,\alpha) \tag{182b}$$

$$g_{12;31}(w,\alpha) = -\frac{1}{16}\sin 2(w-\alpha)\cos(w-\alpha)\left[\cos^2\phi_D \cos^2 w + \sin^2\phi_D\right] ,$$

$$= -\frac{1}{4}\sin 2(w-\alpha)\, g_{12}(w,\alpha) \quad , \tag{182c}$$

$$g_{23;23}(w,\alpha) = \frac{1}{8}\sin^2(w-\alpha)\left[\cos^2\phi_D \cos^2 w + \sin^2\phi_D\right] = \frac{1}{2}\, g_{33}(w,\alpha) \tag{182d}$$

$$g_{23;13}(w,\alpha) = -\frac{1}{16}\ \sin 2(w-\alpha)\ \sin (w-\alpha) \left[\cos^2 \phi_D \cos^2 w + \sin^2 \phi_D\right]$$

$$= -\frac{1}{4}\ \sin 2(w-\alpha)\ g_{32}(w,\alpha)\ , \tag{182e}$$

$$g_{13;13}(w,\alpha) = \frac{1}{32} \sin^2 2(w-\alpha)\ \left[\cos^2 \phi_D \cos^2 w + \sin^2 \phi_D\right]$$

$$= \frac{1}{8} \sin^2 2(w-\alpha)\ g_{22}(w,\alpha)\ . \tag{182f}$$

The other three projection factors can be obtained by an appropriate interchange of indexes.

For collocated sensors $\underline{\rho} = 0$. In this case a completely isotropic internal wave field ensures that the horizontal-vertical gradients 12, 23 are completely decorrelated from the horizontal-horizontal gradient 13. On the other hand, partial correlation exists between the two horizontal-vertical gradients. Thus one finds that the spectral coherence function $\gamma_{12;23}$ is given by the same expression as γ_{13} on page 84.

The general formulas for the temporal cross-spectra for magnetic field gradients are obtained from Eq. (153) and Eq. (154) by simply placing the additional factor $K_n^2(\omega)$ in the numerator of these expressions and replacing $g_{\nu\mu}$ by $2g_{\nu\mu;rs}$. The qualitative aspects of the discussion on pages 84 and 85 apply also to gradients. The numerical values of the maximum directive gain for gradient spectra 12;12 and 23;23 (i.e., horizontal to vertical) are precisely the same as for the horizontal field components. On the other hand, for the gradient spectrum 13;13 (i.e., horizontal to horizontal) the achievable maximum directive gain is somewhat higher. For example, for $\alpha = 0$, $\phi_D = 0$, the gain turns out to be $128/27 \approx 4.74$ or 6.76 dB, which corresponds to the wave direction $w_o = \cos^{-1} \sqrt{\frac{2}{3}}$.

Under the same assumptions as those underlying Eq. (162), the spatial cross-spectra for the magnetic field gradients are

$$S_{\nu\mu;rs}(\underline{K},y) = \frac{C}{2\pi} \frac{(\sigma\mu_o B_p)^2}{1 + 3\cos^2\phi_D} \mathcal{E}_{\nu\mu;rs}(w,\alpha)\, e^{-2Ky} K^{-p+1} \int_{-\infty}^{0} y^2 N^2(y)\, e^{2Ky}\, dy \ . \tag{183}$$

From this we can compute the total r.m.s. gradients above the ocean surface. For each of the gradients we have

$$\langle G^2_{\nu\mu;rs}(y,t)\rangle = \int_0^\infty K\, dK \int_0^{2\pi} dw\, S_{\nu\mu;rs}(\underline{K},y) \ . \tag{184}$$

The total r.m.s. gradient will be defined by

$$G^2_{rms}(y) = \langle G^2_{12;12}\rangle + \langle G^2_{23;23}\rangle + \langle G^2_{13;13}\rangle \ . \tag{185}$$

We call the sum of the first two quantities on the right of Eq. (185) the square of the total r.m.s. horizontal-vertical gradient $G^{(HV)}_{rms}$,

$$G^{2(HV)}_{rms}(y) = \langle G^2_{12;12}(y,t)\rangle + \langle G^2_{23;23}(y,t)\rangle \ . \tag{186}$$

The integrals over the two projection factors entering in Eq. (186) are given by one half of the expressions on the right of Eqs. (165a) and (165c). Summing the two contributions we obtain

$$G^{2(HV)}_{rms}(y) = \frac{C}{16}\left(\sigma\mu_o B_p\right)^2 \frac{1 + 3\sin^2\phi_D}{1 + 3\cos^2\phi_D} \int_0^\infty dK\, e^{-2Ky} K^{-p+2} \int_{-\infty}^{0} y'^2 N^2(y')\, e^{2Ky'}\, dy' , \tag{187}$$

which is very similar to Eq. (166). The essential difference is in the additional factor of K^2 in the integrand. To obtain a numerical estimate of the strength of the r.m.s. gradients, we again consider the simple case of an exponentially stratified ocean and take p = 2. We then find

$$G^{2\,(HV)}_{rms}(y) = \frac{Cb^3N^2(0)}{64}\left(\sigma\mu_o\, B_p\right)^2 \frac{1 + 3\sin^2\phi_D}{1 + 3\cos^2\phi_D}\int_0^\infty dK\, \frac{e^{-2Ky}}{(Kb+1)^3} . \qquad (188)$$

Note that the assumption of a nonzero cutoff wave number K_c is not necessary in this case since the integral is well behaved at the origin. (Cf. Eq. (168).) However, K_c still enters into the problem through its relationship to C, Eq. (E-125). (In fact, $K_c = 0$ will yield C = 0.) At y = 0 the integral in Eq. (188) equals $\frac{1}{2b}$ so that

$$G^{2\,(HV)}_{rms}(0) = \left(\sigma\mu_o\, B_p\right)^2 \frac{1 + 3\sin^2\phi_D}{1 + 3\cos^2\phi_D}\, \frac{E}{128b\pi\rho_o} \cdot \frac{\nu_c}{1 - \nu_c \ln\frac{1+\nu_c}{\nu_c}} . \qquad (189)$$

For the same parameters as employed in Eq. (170), the total r.m.s. horizontal-to-vertical gradient is

$$G^{(HV)}_{rms} \approx \sqrt{\frac{1 + 3\sin^2\phi_D}{1 + 3\cos^2\phi_D}} \cdot 0.655\ \mathrm{pT/m} . \qquad (190)$$

We see that the horizontal-to-vertical r.m.s. magnetic field gradient is largest in the polar regions (1.31 pT/m) and drops 1/4 of this value in the equatorial zone (.33 pT/m).

Although the value of .33 pT/m is not very large, it must be remembered that it is compressed within a bandwidth of about 10^{-3} Hz. If the .33 pT/m were uniformly distributed within this band, one would have a spectral density of about 100 $(\mathrm{pT/m})^2/\mathrm{Hz}$,

which is several orders of magnitude higher than the sensitivity of existing superconducting gradiometers ($\sim .1\ (\mathrm{pT/m})^2/\mathrm{Hz}$). The induced gradient decays above the ocean surface. It may be shown from Eq. (187) that the decay is algebraic, i.e., as $1/y$. This slow decay is due entirely to the fact that the spatial spectral maximum occurs at $K = 0$. For a more realistic assessment we should truncate the lower limit at $K = K_c$, in which case the decay for sufficiently large y will eventually be dominated by $\exp - 2K_c y/b$.

The preceding calculation was carried out for $p = 2$. To assess the sensitivity of the numerical estimate in Eq. (190) to p, we now carry out the calculation for $p = 1$. For this case Eq. (188) is modified in two respects: an additional factor of K appears in the integrand and the formula for C is given by Eq. (E-124). For $y = 0$ the integration yields $1/2b^2$. Employing Eq. (E-124) one finds for $p = 1$

$$G^{2\,(H,V)}_{rms}(0) = (\sigma\mu_0 B_F)^2 \, \frac{1+3\sin^2\phi_D}{1+3\cos^2\phi_D} \, \frac{\Sigma}{128\,\pi\rho_0 b} \cdot \frac{1}{\ln \dfrac{1+\nu_c}{\nu_c}} \, . \tag{191}$$

Comparing this with (189) we observe that the r.m.s. gradient in (191) is larger than that given by (190) by the factor

$$\sqrt{\frac{1 - \nu_c \ln \dfrac{1+\nu_c}{\nu_c}}{\nu_c \cdot \ln \dfrac{1+\nu_c}{\nu_c}}} \, .$$

For $\nu_c = .327$, its value is approximately 1.52. Consequently, for $p = 1$

$$G^{(H,V)}_{rms}(0) \approx \sqrt{\frac{1+3\sin^2\phi_D}{1+3\cos^2\phi_D}} \quad \mathrm{pT/m} \, . \tag{192}$$

Thus, the value of p has a fairly minor effect on the r.m.s. value of the magnetic field gradient. On the other hand, we shall find that the spectral distribution of energy contributing to this r.m.s. gradient is modified substantially by different choices of p, particularly for short spatial wavelengths.

We now compute the horizontal-horizontal r.m.s. gradient given by the square root of the last factor in (185). For this purpose we need the integrated value of the wave number projection factor $g_{13;13}$. From (182f) we find

$$\int_0^{2\pi} g_{13;13}(w,\alpha)\, dw = \frac{\pi}{64}\left[\sin^2\phi_D + \frac{1}{2}\cos^2\phi_D\,(1+2\sin^2\alpha)\right], \tag{193}$$

and with the aid of (183) the square of the r.m.s. value of the horizontal-horizontal gradient becomes

$$G^{2\,(H,H)}_{rms}(y) = \frac{C}{64}\cdot(\sigma\mu_o B_p)^2\,\frac{2\sin^2\phi_D + \cos^2\phi_D\,(1+2\sin^2\alpha)}{4\,(1+3\cos^2\phi_D)}$$
$$\cdot\int_0^{\infty} dK e^{-2Ky}\, K^{-p+2}\int_{-\infty}^{0} y' H^2(y') e^{2Ky'}\, dy' \quad . \tag{194}$$

Comparing this expression with (187), we note that the H-H gradient is smaller than the total H-V gradient. For example, in the equatorial zone ($\phi_D = 0$) we find,

$$\frac{G^{(H,H)}_{rms}(y)}{G^{(H,V)}_{rms}(y)} = \frac{1}{4}\sqrt{1+2\sin^2\alpha} \quad , \tag{195}$$

while in the polar region ($\phi_D = \pi/2$)

$$\frac{G^{(H,H)}_{rms}(y)}{G^{(H,V)}_{rms}(y)} = \frac{1}{32} \quad . \tag{196}$$

With excitation function $I(\underline{K}) = CK^{-p}$, the temporal spectra for the three principal gradients can be written in a form similar to (175):

$$\Phi_{12;12}(\omega,y) = \frac{\sin^2\phi_D + \frac{1}{8}\cos^2\phi_D\,(1+2\cos^2\alpha)}{4[1+3\cos^2\phi_D]}\, S_g(\omega,y)\,, \quad (197a)$$

$$\Phi_{23;23}(\omega,y) = \frac{\sin^2\phi_D + \frac{1}{8}\cos^2\phi_D\,(1+2\sin^2\alpha)}{4[1+3\cos^2\phi_D]}\, S_g(\omega,y)\,, \quad (197b)$$

$$\Phi_{13;13}(\omega,y) = \frac{\sin^2\phi_D + \frac{1}{2}\cos^2\phi_D\,(1+2\sin^2\alpha)}{64[1+3\cos^2\phi_D]}\, S_g(\omega,y)\,, \quad (197c)$$

where $S_g(\omega,y)$ is the normalized gradient spectrum given by

$$S_g(\omega,y) = 2\pi C\,(\sigma\mu_o B_p)^2\,\omega^4 \sum_n e^{-2K_n(\omega)y}\,\frac{\left[K_n(\omega)\right]^{-p+1} L_n^2[K_n(\omega)]}{K_n(\omega)\,v_{gn}(\omega)}\,. \quad (198)$$

For an exponentially stratified ocean, the normalized gradient spectrum follows from the normalized component spectrum in (178) by simply multiplying each term in the series by $K_n(\omega) \equiv \nu_n^2/b^2$. Consequently,

$$S_g(y,\omega) = 4\pi(\sigma\mu_o B_p)^2\, CN(0)b^p \sum_{n=1}^{\infty} \nu_n^2\, s_n^{(p)}(\eta,y)\,, \quad (199)$$

with $s_n^{(p)}(\eta,y)$ given by (179). For $b = 1300$ m, the constant in front of the sum becomes

$$4\pi(\sigma\mu_o B_p)^2 Cb^p N(0) = \begin{cases} 1.331 \times 10^5 \left(\frac{pT}{m}\right)^2/\mathrm{Hz}\ ; \ p = 2\,, \\ 1.576 \times 10^5 \left(\frac{pT}{m}\right)^2/\mathrm{Hz}\ ; \ p = 1\,. \end{cases}$$

A plot of the spectrum of the horizontal-vertical gradient, ($\alpha = 0$, $\phi_D = 0$), Eq. 197a), is shown in Fig. 7. Figure 8 shows the normalized gradient spectrum $S_g(y,\omega)$. These calculations have been carried out for the same internal wave physical parameters as the component spectra in Figs. 5 and 6. One observes that the differences between the gradient spectra for the case p = 2 and p = 1 are quite pronounced, especially near the ocean

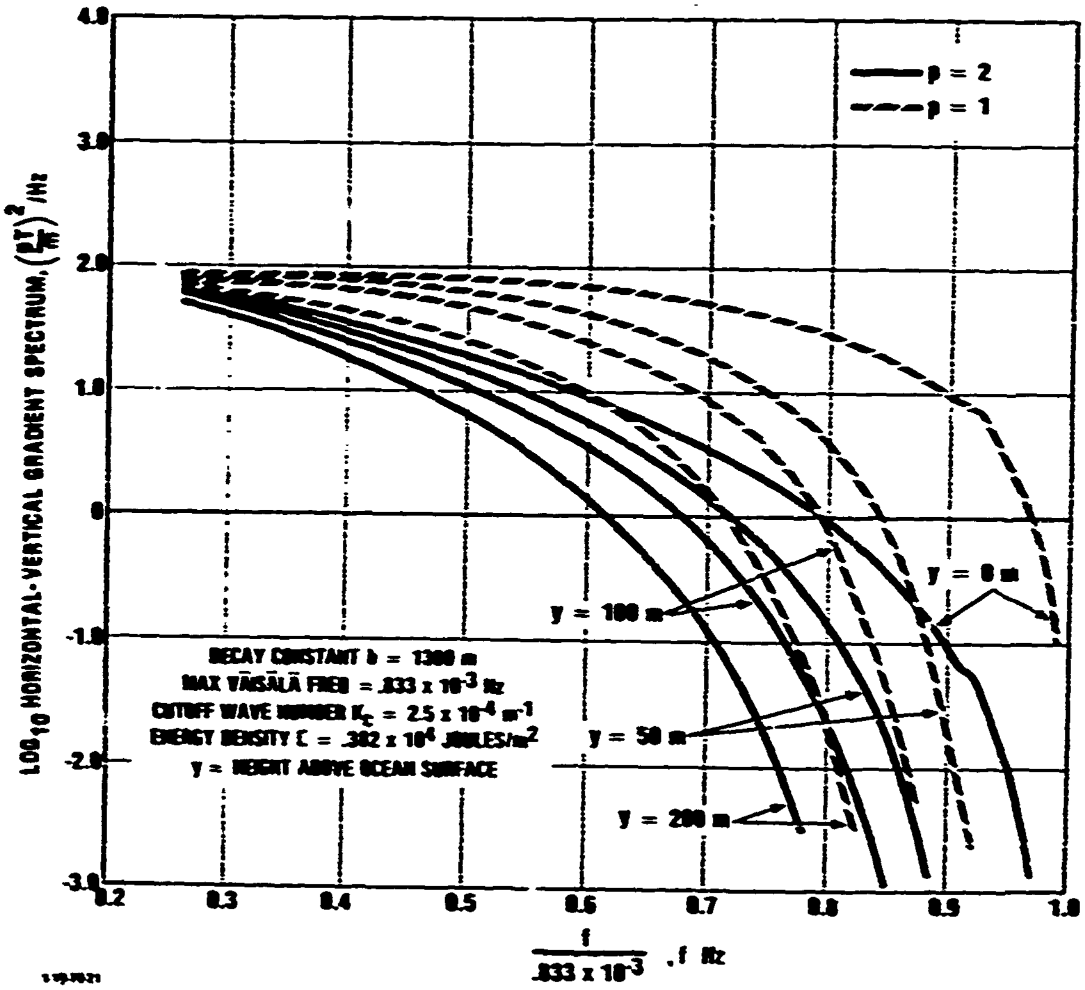

FIGURE 7. Horizontal-to-vertical gradient of magnetic field induced by internal waves in an exponentially stratified ocean ($\alpha = 0$, $\phi_D = 0$)

surface. On the other hand, it will be recalled (cf. Eqs. 190 and 192) that difference in levels of the r.m.s. gradients for p = 1 and p = 2 is rather insignificant (a factor of about 1.5). Evidently, most of the integrated contribution to these r.m.s. values arises from extremely low frequencies (and long wavelengths). This is quite compatible with the curves in Figs. 7 and 8, all of which merge toward the lower frequency band edge.

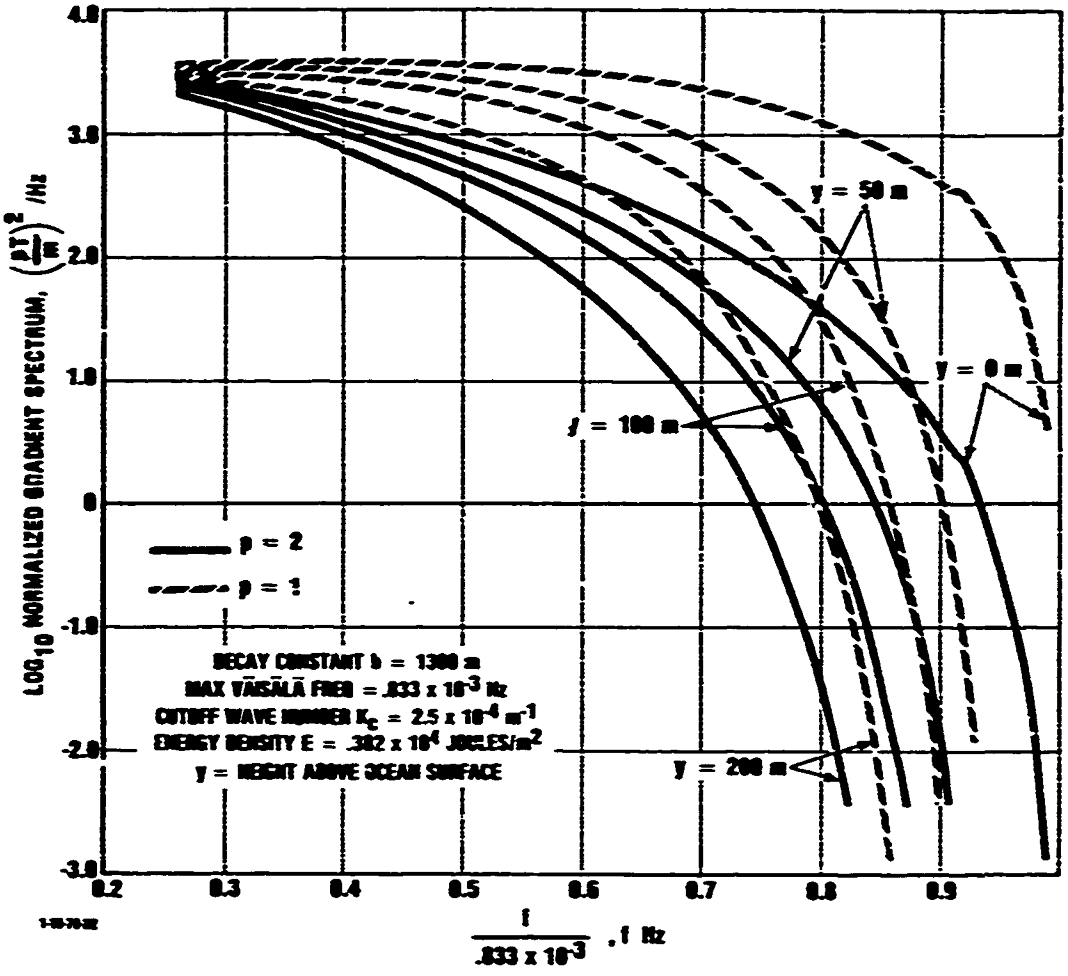

FIGURE 8. Normalized gradient of magnetic field induced by internal waves in an exponentially stratified ocean

C. SURFACE-WAVE-INDUCED MAGNETIC FIELD SPECTRA ABOVE THE OCEAN SURFACE

The formal procedure leading to the expressions for the correlation functions and spectra of surface-wave-induced electromagnetic fields is very similar to that employed in the preceding for internal-wave-induced magnetic fields. For the magnetic field components we use Eq. (131) and obtain the correlation between components ν and μ measured at the same height above the ocean:

$$R_{\nu\mu}(\underline{\rho}',\underline{\rho}'',t',t'';y) = \frac{(\sigma\mu_o B_p)^2}{4(1+3\cos^2\phi_p)}$$

$$\iint_{-\infty}^{\infty} d\underline{K}' \iint_{-\infty}^{\infty} d\underline{K}''\, e^{-i\underline{K}'\cdot\underline{\rho}'+i\underline{K}''\cdot\underline{\rho}''}\, e^{-(K'+K'')y}\, g_\nu(w',\alpha)\, g_\mu^*(w'',\alpha)\, \langle h_s(\underline{K}',t')\, h_s^*(\underline{K}'',t'')\rangle \,. \tag{200}$$

The wave number projection factors $g_\nu g_\mu^*$ are given by Eq. (148). The statistical average in the integrand is evaluated by reference to Eq. (129) and Eq. (A-26) in Appendix A:

$$\langle h_s(\underline{K}',t')\, h_s^*(\underline{K}'',t'')\rangle$$

$$= \frac{g}{2K'}\left[\psi(\underline{K}')\, e^{i\Omega(K')(t'-t'')} + \psi(-\underline{K}')\, e^{-i\Omega(K')(t'-t'')}\right]\delta(\underline{K}'-\underline{K}'') \,. \tag{201}$$

Consequently, with $\underline{\rho}' - \underline{\rho}' = \underline{\rho}$, $t' - t'' = \tau$, Eq. (200) becomes

$$R_{\nu\mu}(\underline{\rho},\tau,y) = \frac{g}{8}\cdot\frac{(\sigma\mu_o B_p)^2}{1+3\cos^2\phi_D} \cdot$$

$$\iint_{-\infty}^{\infty} d^2\underline{K}\, e^{-i\underline{K}\cdot\underline{\rho}}\, e^{-2Ky}\, K^{-1}\, g_{\nu\mu}(w,\alpha)\left[\psi(\underline{K})\, e^{i\Omega(K)\tau} + \psi(-\underline{K})\, e^{-i\Omega(K)\tau}\right] . \qquad (202)$$

Taking account of the dispersion relation $\Omega^2 = Kg$ the Fourier transform of Eq. (202) with respect to τ yields (for $\omega > 0$).

$$\Phi_{\nu\mu}(\underline{\rho},\omega,y) = \frac{\pi}{2}\cdot\frac{(\sigma\mu_o B_p)^2}{1+3\cos^2\phi_D}$$

$$\omega\, e^{-2y\frac{\omega^2}{g}}\int_0^{2\pi} dw\, e^{-i\frac{\omega^2}{g}\rho\cos(w-\theta)}\, g_{\nu\mu}(w,\alpha)\, \psi\left(\frac{\omega^2}{g},w\right) , \qquad (203)$$

which for collocated component sensors becomes

$$\Phi_{\nu\mu}(0,\omega,y) = \frac{\pi}{2}\cdot\frac{(\sigma\mu_o B_p)^2}{1+3\cos^2\phi_D}\quad \omega\, e^{-2y\frac{\omega^2}{g}}\int_0^{2\pi} dw\, g_{\nu\mu}(w,\alpha)\, \psi\left(\frac{\omega^2}{g},w\right) . \qquad (204)$$

With the Pierson Neumann spectrum for $\psi\left(\frac{\omega^2}{g},w\right)$, Eq. (A-38), substituted in Eq. (204) we obtain

$\Phi_{\nu\mu}(0,\omega,y) =$

$$\frac{\pi\bar{C}g^2}{8} \cdot \frac{(6\mu_o B_p)^2}{1+3\cos^2\phi_D}\, \omega^{-8} e^{-2y\frac{\omega^2}{g} - 2g^2\omega^{-2}U^{-2}} \int_{w_o-\pi/2}^{w_o+\pi/2} g_{\nu\mu}(w,\alpha) \cos^2(w-w_o)\, dw \, , \quad (205)$$

where w_o is the wind direction, U wind speed in m/sec, and $\bar{C} = 3.05\ m^2\ sec^{-5}$. As written, Eq. (205) holds for a fully developed sea. In other cases the spectrum must be truncated below the angular frequency $\omega = \omega_I$, the value of which is determined by fetch and wind duration [11]. For a fully developed sea the spectrum is of the form $\omega^{-8} \exp[-\alpha\omega^2 - \beta/\omega^2]$ for $0 < \omega < \infty$, which has a maximum at

$$\omega_{max} = \left(\frac{2}{\alpha}\right)^{\frac{1}{2}} \left[\left(1 + \frac{\alpha\beta}{4}\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}} .$$

Since $\alpha = 2y/g$, $\beta = 2g^2/U^2$, (206)

$$\omega_{max}(y) = \left(\frac{g}{y}\right)^{\frac{1}{2}} \left[\left(\frac{gy}{U^2} + 1\right)^{\frac{1}{2}} - 1\right]^{\frac{1}{2}} . \quad (207)$$

At the ocean surface this becomes

$$\omega_{max}(0) = \frac{1}{\sqrt{2}} \frac{g}{U} .$$

On the other hand, for $\omega_I = 0$, the spectrum of the surface wave displacement decays as ω^{-6} (Eqs. A-34 and A-30) and can be shown to have a maximum at $\omega = \sqrt{\frac{2}{3}}\, g/U$. Thus, the peak of the magnetic field spectrum is shifted in the direction of lower frequencies by about 15 percent. At high elevations above the ocean surface

$$\omega_{max}(\infty) \sim \left(\frac{g}{y}\right)^{\frac{1}{2}} \left(\frac{g}{U}\right)^{\frac{1}{2}} ,$$

so that

$$\frac{\omega_{max}(\infty)}{\omega_{max}(0)} = \sqrt{2}\, \frac{U^{\frac{1}{2}}}{(gy)^{\frac{1}{4}}} ,$$

which shows that for large y the spectral peak shifts even further toward lower frequencies.

The r.m.s. value of the total field is found from

$$B^2_{rms}(y) = \sum_{\ell=1}^{3} 2 \int_0^{\infty} \Phi_{\nu\mu}(0,\omega,y)\, d\omega . \qquad (208)$$

The factor of 2 is used to account for the fact that Eq. (205) represents a doublesided spectral density. After summing Eq. (205) over the three components and carrying out the integration over w, one obtains

$$B^2_{rms}(y) = \frac{\pi^2 \bar{C}\, g^2}{64}\,(\sigma\mu_o\, B_p)^2\, \frac{1+3\sin^2\phi_D + 2\cos^2\phi_D \cos^2 w_o}{1+3\cos^2\phi_D}$$

$$\cdot \int_0^\infty d\omega\, \omega^{-8} \exp\left\{-2y\frac{\omega^2}{g} - 2g^2\omega^2 U^{-2}\right\} \cdot \qquad (209)$$

It may be shown that

$$\int_0^\infty d\omega\, \omega^{-8} \exp\left\{-\alpha\omega^2 - \beta/\omega^2\right\} = \frac{\sqrt{\pi}}{2}\left\{\frac{15}{8}\beta^{-7/2} + \frac{19}{4}\alpha^{1/2}\beta^{-3} + \frac{7}{2}\alpha\beta^{-5/2} + \alpha^{3/2}\beta^{-2}\right\} \exp{-2\sqrt{\alpha\beta}} \,. \qquad (210)$$

With the aid of this formula we rewrite Eq. (209) in the following form:

$$B^2_{rms}(y) \doteq 4\, \frac{1+3\sin^2\phi_D + 2\cos^2\phi_D \cos^2 w_o}{1+3\cos^2\phi_D}\, B^2_p \cdot \xi^2(y)\,, \qquad (211)$$

where ξ is the dimensionless quantity {c.f Ref. [6]}

$$\xi^2(y) =$$

$$\frac{\pi^2 g^2 \bar{C}\, \sigma^2 \mu_o^2}{4096} \cdot \sqrt{\frac{\pi}{2}} \left\{\frac{15}{8}\left(\frac{g}{U}\right)^{-7} + \left(\frac{y}{g}\right)^{1/2}\left(\frac{g}{U}\right)^{-6} + 14\left(\frac{y}{g}\right)\left(\frac{g}{U}\right)^{-5} + 8\left(\frac{y}{g}\right)^{3/2}\left(\frac{g}{U}\right)^{-4}\right\}$$

$$\cdot \exp\left\{-4\,\frac{g}{U}\sqrt{\frac{y}{g}}\right\} \,. \qquad (212)$$

In the equatorial region, we have

$$B_{rms}(y) = B_p\sqrt{1 + 2 \cos^2 w_o}\ \xi(y) ,$$

while in the polar regions, $B_{rms}(y) = 4\ B_p\ \xi(y)$. At the ocean surface Eq. (212) yields

$$\xi(0) = 2.197 \times 10^{-3}\ U^{7/2} , \tag{213}$$

where U is in meters/sec). A plot of Eq. (213) is shown in Fig. 9. To gain an appreciation of the numerical values of $\xi(0)$ consider the equatorial region. There $B_{rms}(0)$ reaches a maximum value of $\approx 1.08 \times 10^8\ \xi(0)$ pT (for $w_o = 0$). Referring to values of $\xi(0)$ in Fig. 9, we find that $B_{rms}(0)$ ranges from the high value of 8.64×10^3 pT for a wind speed of 20 m/sec to the low value of about .4 pT for a wind speed of only 1 m/sec. It is of interest to observe that the high value of the r.m.s. field is of the same order of magnitude as the total r.m.s. field induced by internal waves given by Eq. (170).

Above the ocean surface $\xi(y)$ decays in accordance with Eq. (212). Figure 10 shows a plot of $\xi(y)$ as a function of height above the ocean with the wind speed as a parameter. The lowest value of the ordinate corresponds to r.m.s. field levels of about .1 pT. Note the sensitivity to wind speed of the location of this threshold above the ocean surface. For example, an increase in wind speed from 10 m/sec to only 12.5 m/sec increases the height at which this threshold level is reached from about 400m to 700m.

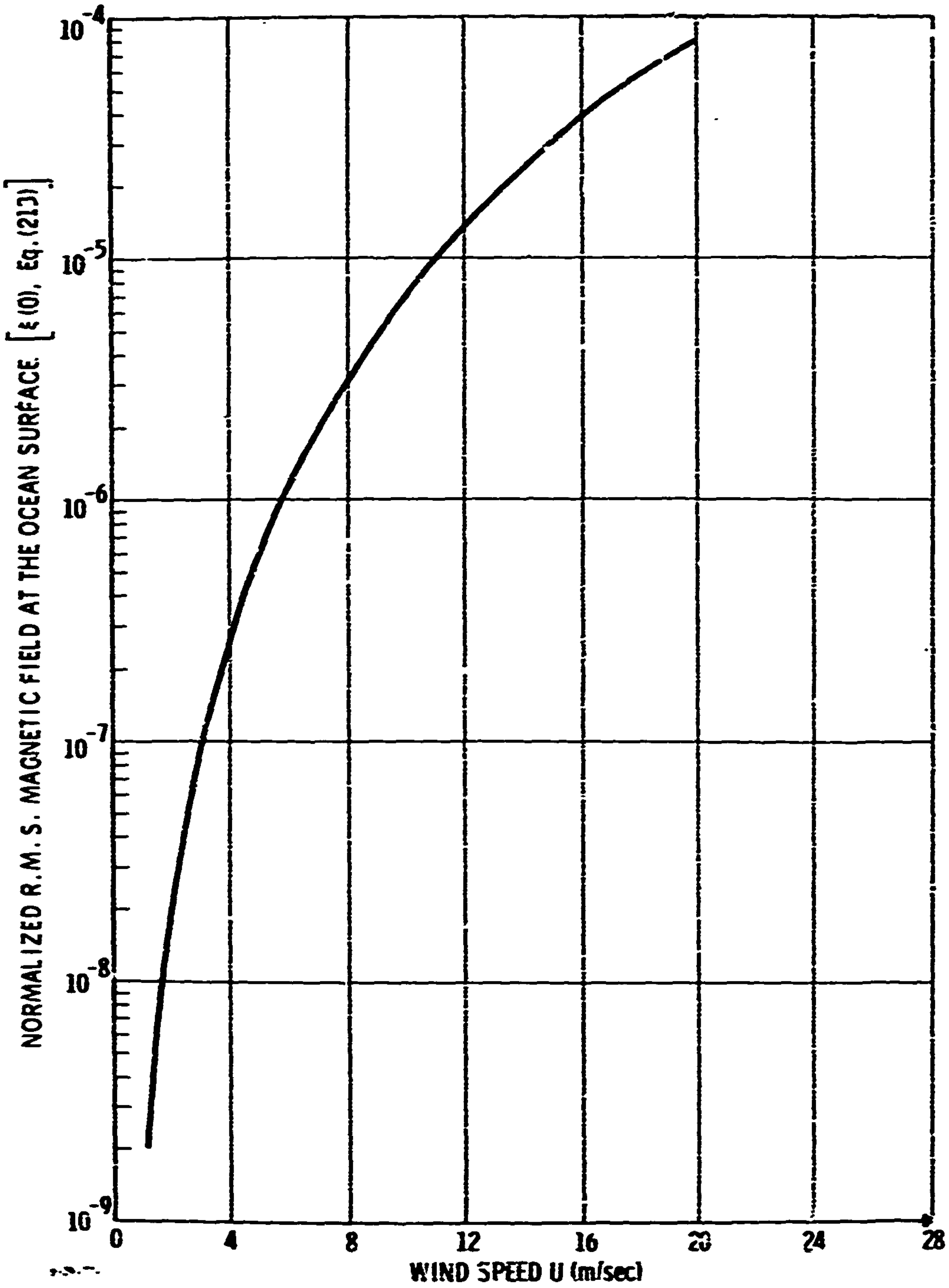

FIGURE 9. Surface-wave-induced (normalized) r.m.s. magnetic field at ocean surface as a function of wind speed

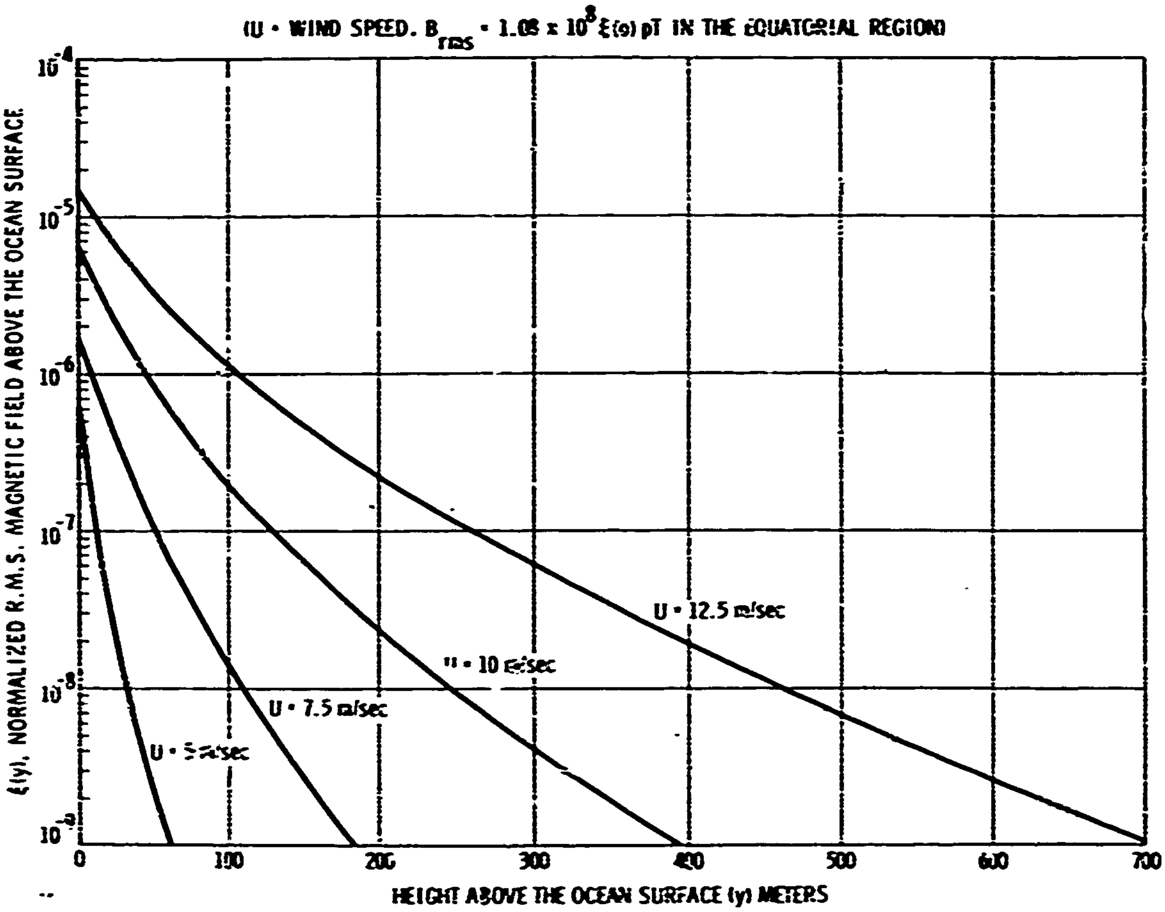

FIGURE 10. Surface-wave-induced (normalized) magnetic field as a function of height above the ocean surface

D. SPECTRA OF SURFACE-WAVE-INDUCED ELECTRIC FIELD COMPONENTS ABOVE THE OCEAN SURFACE

The correlation function among the three mutually perpendicular electric field components E_1, E_2, E_3 is obtained from Eq. (136) through Eq. (138), by analogy with Eq. (200) and Eq. (201). Thus, one finds

$$R_{\nu\mu}^{(E)}(\underline{\rho},\tau,y) = \frac{g\, B_p^{\,2} \cos^2 \phi_D}{2(1+ 3 \cos^2 \phi_D)}$$

$$\cdot \iint\limits_{-\infty}^{\infty} d^2\underline{K}\, e^{-i\underline{K}\cdot\underline{\rho}}\, e^{-2Ky}\, K\, g_{\nu\mu}^{(E)}(w,\alpha) \left[\psi(\underline{K})\, e^{i\Omega(K)\tau} + \psi(-\underline{K})\, e^{-i\Omega(K)\tau}\right]. \qquad (214)$$

$$g_{\nu\mu}^{(E)} = g_{\nu}^{(E)}\, g_{\mu}^{*(E)} \qquad (215)$$

$$g_1^{(E)}(w,\alpha) = \cos(w-\alpha) \sin w \;, \qquad (216a)$$

$$g_2^{(E)}(w,\alpha) = -\, i \sin w \;, \qquad (216b)$$

$$g_3^{(E)}(w,\alpha) = \sin(w-\alpha) \sin w \;. \qquad (216c)$$

The temporal spectra for $\underline{\rho} = 0$ become

$$\Phi_{\nu\mu}^{(E)}(0,\omega,y) = \frac{2\pi\, B_p^{\,2} \cos^2 \phi_D}{g^2\left(1 + 3 \cos^2 \phi_D\right)}\, \omega^5\, e^{-2y\frac{\omega^2}{g}} \int_0^{2\pi} dw\, g_{\nu\mu}^{(E)}(w,\alpha)\, \psi\!\left(\frac{\omega^2}{g}, w\right) . \qquad (217)$$

Comparing this expression with the spectrum for the magnetic field components, Eq. (204), we observe that the two spectra are essentially identical except for the presence of an additional factor of ω^4 in the electric field spectrum. Therefore, at $y = 0$, the relative high frequency spectral constituents in

the electric field will be larger, and the eventual decay rate of the electric field spectrum with increasing frequency correspondingly slower, than for the magnetic field. It is important to note that for $\nu = \mu$, Eq. (217) gives the spectra of the individual electric field components and that the sum of these spectra is not equal to the spectrum of the magnitude of the total electric field. The computation of the spectrum of the latter involves the same difficulties as mentioned in connection with Eq. (174) for the total magnetic field.

After the expression in Eq. (A-30) for the Pierson-Neumann surface wave spectrum is substituted in Eq. (217) one obtains

$$\Phi_{\nu\mu}^{(E)}(0,\omega,y) = \frac{\pi C\, B_p^2 \cos^2 \phi_D}{2(1 + 3\cos^2 \phi_D)}$$

$$\cdot\ \omega^{-4} e^{-2y\frac{\omega^2}{g} - 2g^2\omega^{-2}U^{-2}} \int_{w_o-\pi/2}^{w_o+\pi/2} dw\, g_{\nu\mu}^{(E)}(w,\alpha)\cos^2(w-w_o)\ . \tag{218}$$

The total r.m.s. electric field is now computed from the expression

$$E_{rms}^2(y) = 2\sum_{\nu=1}^{3} \int_0^{\infty} \Phi_{\nu\mu}(0,\omega,y)\, d\omega\ . \tag{219}$$

From Eq. (216) one finds

$$\sum_{\nu=1}^{3} g_{\nu\mu}^{(E)}(w,\alpha) = 2\sin^2 w\ ,$$

also

$$\int_{w_0-\pi/2}^{w_0+\pi/2} 2\sin^2 w \cos^2(w-w_0)\,dw = \frac{\pi}{4}\left(1 + 2\sin^2 w_0\right) .$$

We then obtain

$$E^2_{rms}(y) = \pi^2 C \; B_p^2 \, \frac{\cos^2 \phi_D \left(1 + 2\sin^2 w_0\right)}{4\left(1 + 3\cos^2 \phi_D\right)} \int_0^\infty d\omega\, \omega^{-4} \exp\left\{-2y\frac{\omega^2}{g} - 2g^2\omega^{-2}U^{-2}\right\} . \tag{220}$$

At $y = 0$ we can use the formula

$$\int_0^\infty \omega^{-4}\, e^{-\beta\omega^{-2}}\, d\omega = \frac{\sqrt{\pi}}{4}\,\beta^{-3/2} .$$

With $\beta = 2g^2U^{-2}$ this yields $\sqrt{\pi}\; 2^{-7/2}(U/g)^3$. Substituting this in Eq. (220) together with the numerical factors we obtain the simple formula

$$E_{rms}(0) \cong 2.21 \,\frac{\sqrt{1 + 2\sin^2 w_0}\,\cos\phi_D}{\sqrt{1 + 3\cos^2\phi_D}}\, U^{3/2} \;\;\mu\text{volts/m} , \tag{221}$$

where the wind speed U is in meters/sec. Thus, the r.m.s. electric field at the ocean surface increases only as the $3/2$ power of the wind speed, whereas the r.m.s. magnetic field was found to have a $U^{7/2}$ dependence (Eq. (213)). The dependence on the wind direction w_0 (measured relative to the vertical plane containing the geomagnetic field) is weak, as was also found to be the case for the r.m.s. magnetic field (Eq. (211)). The particular functional dependence is, of course, a consequence of the assumed $\cos^2(w-w_0)$ directionality of the surface wave spectrum. A surface wave spectrum with a greater degree of directionality will result in a larger variation of the induced electric field with w_0. In particular, for a perfectly unidirectional surface wave train, the induced field will

vary from its maximum at $w_o = \pi/2$ (wind direction perpendicular to the vertical plane containing the geomagnetic field), down to zero for $w_o = 0$ (or, π)(wind direction in the plane containing the geomagnetic field). On the other hand, in Eq. (221) the total excursion of the r.m.s. field as w_o is varied from 0 to $\pi/2$ is only $\sqrt{3}$.

The magnitude of the r.m.s. electric field as predicted by Eq. (221) is certainly of sufficiently high level so as to be measurable in the absence of other competing noise sources. For example, for $U = 10$ m/sec, $\phi_D = 0$, $w_o = \pi/2$, one obtains $E_{rms}(0) \cong 60$ μvolts/m.

E. SPECTRA OF SURFACE-WAVE-INDUCED MAGNETIC FIELD GRADIENTS

The correlation functions for surface-wave-induced gradients of the magnetic field can be derived with the aid of Eq. (132) and Eq. (201). Alternatively, by recognizing that the formal relation between field components and their gradients does not depend on whether these quantities are induced by internal or surface waves, we can employ the results of VI-A, and thus obtain these correlation functions directly from Eq. (202) by replacing $g_{\nu\mu}$ by $g_{\nu\mu;rs}$ and supplying the additional factor K^2 in the integrand. In any case, there results

$$R_{\nu\mu;rs}(\underline{\rho},\tau,y) = \frac{1}{4} \frac{g\,(\sigma\mu_o\, B_p)^2}{\left(1 + 3\cos^2 \phi_D\right)}$$

$$\iint_{-\infty}^{\infty} d^2\underline{K}\, e^{-i\underline{K}\cdot\underline{\rho}}\, e^{-2Ky}\, K\, g_{\nu\mu;rs}(w,\alpha)\left[\psi(\underline{K})\, e^{i\Omega(K)\tau} + \psi(-\underline{K})\, e^{-i\Omega(K)\tau}\right]. \qquad (222)$$

The corresponding temporal spectra are:

$$\Phi_{\nu\mu;rs}(\underline{\rho},\omega,y)$$

$$= \frac{\pi\,(\sigma\mu_o\,B_p)^2}{g^2(1+3\cos^2\phi_D)}\;\omega^5\;e^{-2y\frac{\omega^2}{g}}\int_0^{2\pi} dw\; e^{-i\frac{\omega^2}{g}\rho\cos(w-\theta)}\,g_{\nu\mu;rs}(w,\alpha)\;\psi(\frac{\omega^2}{g},w)\,, \tag{223}$$

and

$$\Phi_{\nu\mu}(0,\omega,y)$$

$$= \frac{\pi\,(\sigma\mu_o\,B_p)^2}{g^2(1+3\cos^2\phi_D)}\;\omega^5\;e^{-2y\frac{\omega^2}{g}}\int_0^{2\pi} dw\;g_{\nu\mu;rs}(w,\alpha)\;\psi(\frac{\omega^2}{g},w)\quad. \tag{224}$$

Thus, the dependence on frequency of the gradient spectra is exactly the same as that of the electric field component spectra, Eq. (217). For the Pierson-Neumann spectrum Eq. (224) reads.

$$\Phi_{\nu\mu;rs}(0,\omega,y) = \pi\,\overline{C}\,\frac{(\sigma\mu_o\,B_p)^2}{4(1+3\cos^2\phi_D)}\,\omega^{-4}\,e^{-2y\frac{\omega^2}{g}-2g^2\omega^{-2}U^{-2}}\int_{\pi/2-w_o}^{\pi/2+w_o} dw\;g_{\nu\mu;rs}(w,\alpha)\,\cos^2(w-w_o) \tag{225}$$

The spectrum attains its maximum value at

$$\omega_{max} = (\frac{g}{2y})^{\frac{1}{2}}\cdot\left[(1+\frac{2gy}{U^2})^{\frac{1}{2}}-1\right]^{\frac{1}{2}}\quad. \tag{226}$$

For $y = 0, \omega_{max} = \frac{g}{U}$, which is about 20 percent higher than the spectral maximum of the ocean surface displacement.

We now compute the r.m.s. horizontal-vertical gradient, defined as in Eq. (186). We have

$$G^{2\,(HV)}_{rms}(y) = 2\int_0^{\infty} d\omega\left[\Phi_{12;12}(\omega,y)+\Phi_{23;23}(\omega,y)\right]\,.$$

After substituting from Eq. (225), the integration over w is carried out with the aid of the following formulae:

$$\int_{-\pi/2}^{\pi/2} dw \cos^2(w-w_o)\left[g_{12;12}(w,\alpha) + g_{13;13}(w,\alpha)\right] dw$$

$$= \frac{1}{8}\int_{-\pi/2}^{\pi/2} \left[\cos^2 w \cos^2 \phi_D + \sin^2 \phi_D\right] \cos^2(w-w_o)\, dw$$

$$= \frac{\pi}{64}\left[\cos^2 \phi_D (1 + 2\cos^2 w_o) + 4\sin^2 \phi_D\right] .$$

One then obtains

$$G^{2(HV)}_{rms}(y)$$

$$= \frac{\pi^2}{32}\,\overline{C}\,(\sigma\mu_o B_p)^2 \frac{\left[\cos^2 \phi_D (1 + 2\cos^2 w_o) + 4\sin^2 \phi_D\right]}{4(1 + 3\cos^2 \phi_D)} \int_0^\infty d\omega\, \omega^{-4}\, e^{-2y\frac{\omega^2}{g} - 2g^2\omega^{-2}U^{-2}} \tag{227}$$

For y = 0 we use the integration formula following Eq. (220). After substituting numerical factors in Eq. (227), the r.m.s. gradient at the ocean surface becomes

$$G^{(HV)}_{rms}(0) \cong 3.925 \sqrt{\frac{\cos^2 \phi_D (1 + 2\cos^2 w_o) + 4\sin^2 \phi_D}{4(1 + 3\cos^2 \phi_D)}}\; U^{3/2}\ \mathrm{pT/m}\ . \tag{228}$$

For $\phi_D = 0$, $w_o = 0$ and U = 10 m/sec one finds $G^{(HV)}_{rms} \approx 53.7$ pT/m.

VII. MAGNETIC FIELD SPECTRA OBSERVED FROM MOVING MEASUREMENT PLATFORMS

Open ocean measurements of magnetic field and gradient spectra must invariably contend with platform motion relative to the geostationary coordinate system. Consequently, the ideal spectra discussed in the preceding section would, in general, not be directly observable. Platform motion may, of course, also be introduced deliberately to increase the area of ocean surface traversed per unit time, as, e.g., in magnetic anomaly detection from an aircraft. For purposes of analysis, it is convenient to distinguish between the steady or systematic motion, and fluctuations in time of the mean position of the measurement platform, which fluctuations are attributable to imperfect platform stability. While of great practical importance, questions of stability cannot sensibly be addressed without recourse to data relating to specific platforms. We shall, therefore, restrict the subsequent discussion to surface-wave-induced and internal-wave-induced magnetic field spectra as they are modified by the introduction of a steady component of platform motion.

A. SURFACE-WAVE-INDUCED MAGNETIC FIELD AND GRADIENT SPECTRA OBSERVED FROM A MOVING PLATFORM

We assume a uniform velocity $\underline{V}$ parallel to the ocean surface. The velocity vector $\underline{V}$ is oriented along the unit vector $\underline{\ell}_1$ in Fig. 3; the angle α is now a measure of the relative orientation of the direction of platform motion and a vertical plane parallel to the geomagnetic field. The induced magnetic field is again resolved along the coordinate axes $\underline{\ell}_1$, $\underline{\ell}_2$, $\underline{\ell}_3$:

component 1 is resolved along the "track," while $\underline{\ell}_3$ corresponds to the "cross-track" component.

For collocated sensors the correlation function between components $\nu\mu$ measured in the moving platform coordinates can be obtained from (202) by a replacement of the transverse coordinate $\underline{\rho}$ in the integrand by $\underline{K}.\underline{V}\ \tau$. The elements of the temporal spectral matrix are found by taking the Fourier transform with respect to τ:

$$\Phi^{(V)}_{\nu\mu}(\omega,y) = \frac{\pi g}{4} \frac{(\sigma\mu_0 B_p)^2}{1+3\cos^2\phi_D} \cdot$$

$$\iint_{-\infty}^{\infty} d^2\underline{K}\ e^{-2Ky} K^{-1} \varepsilon_{\nu\mu}(w,\alpha) \left\{\psi(\underline{K})\delta\ [\Omega(K)-\underline{K}.\underline{V}-\omega] + \psi(-\underline{K})\ \delta[\Omega(K) + \underline{K}.\underline{V} + \omega]\right\} . \quad (229)$$

We shall express the variables of integration in polar form and carry out the integration first with respect to K. Contributions from the two delta functions are obtained if

$$\omega + \underline{K} \cdot \underline{V} - \Omega(K) = 0 \quad , \qquad (230a)$$

$$\omega + \underline{K} \cdot \underline{V} + \Omega(K) = 0 \quad . \qquad (230b)$$

Since $\Omega(K) = + \sqrt{Kg}$, the two preceding equations are equivalent to

$$(\omega + vK)^2 = Kg, \qquad (231)$$

where

$$v = V\cos\ (\alpha - w) \ , \qquad (232)$$

is the projection of the platform velocity vector on $\underline{K}$ (see Fig. 11).

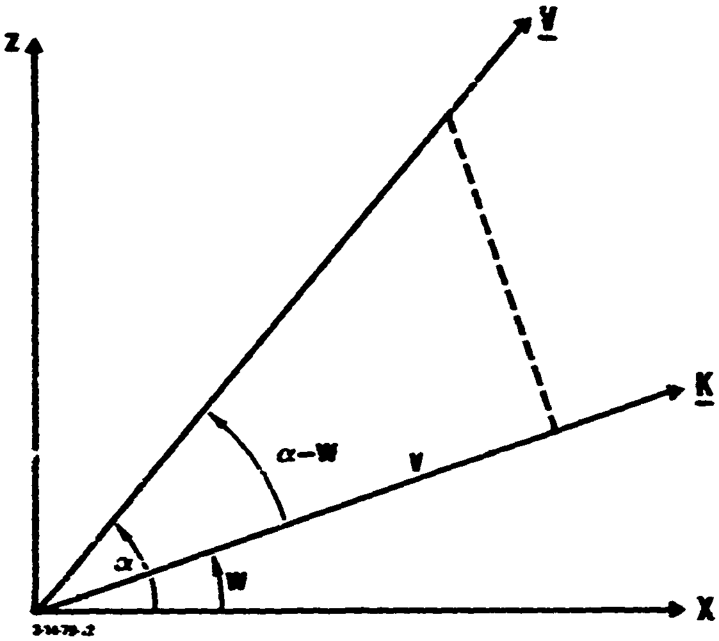

FIGURE 11.

The two solutions of (231) for K will be denoted by K_1 and K_2. They may be expressed in the following form

$$K_{1,2} = \frac{\omega^2}{g} f_{1,2}(\delta) \quad , \tag{233}$$

where the new roots f_1, f_2 satisfy

$$\delta^2 f_{1,2}^2 + (2\delta - 1) f_{1,2} + 1 = 0 \quad , \tag{234}$$

and the dimensionless variable δ is

$$\delta = \frac{V\omega}{g} \quad . \tag{235}$$

The two solutions of (234) are

$$f_1(\delta) = \frac{1-2\delta + \sqrt{1-4\delta}}{2\delta^2} \quad , \tag{236}$$

$$f_2(\delta) = \frac{1-2\delta - \sqrt{1-4\delta}}{2\delta^2} \quad . \tag{237}$$

Evidently,

$$\delta < \frac{1}{4} \quad , \tag{238}$$

since only real roots are of interest. A plot of f_1 and f_2 is shown in Fig. 12.

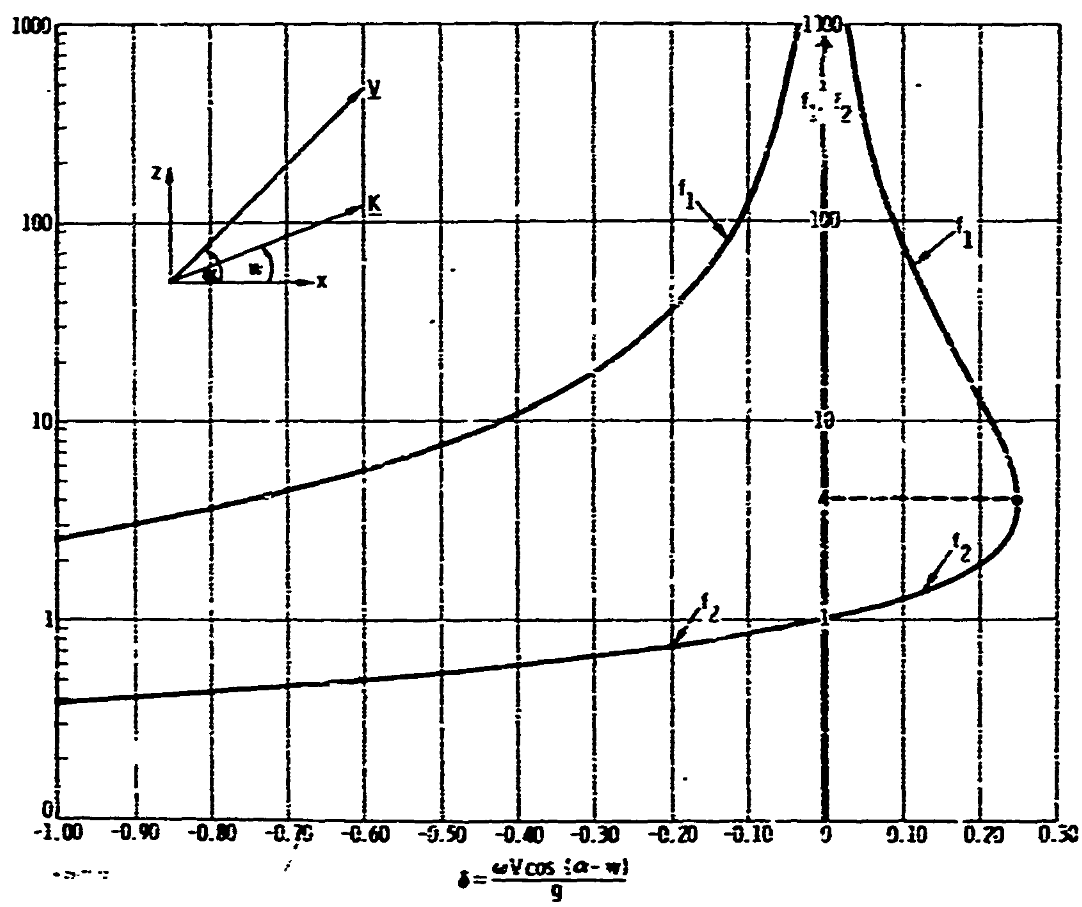

FIGURE 12.

For small δ one finds the following limiting forms:

$$f_1(\delta)_{\delta\sim 0} \sim \frac{1}{\delta^2} \equiv \left(\frac{g}{\omega}\right)^2 \frac{1}{v^2} \ , \quad f_2(\delta)_{\delta\sim 0} \sim 1 \ . \tag{239}$$

The corresponding propagation constants are

$$K_1 \sim \frac{g}{v^2} = \frac{g}{V^2\cos^2(\alpha - w)} \ , \quad K_2 \sim \frac{\omega^2}{g} \ . \tag{240}$$

These limiting forms are valid for zero platform velocity; K_2 is seen to approach the surface wave propagation constant corresponding to a stationary platform, while K_1 tends to infinity. The contribution from the latter tends to zero since the integrand decays with increasing wave numbers.

We now consider the individual contributions from each of the two delta functions (229). Since $\Omega = \sqrt{Kg} > 0$, there will be contributions from (230a) only if the radian frequency ω falls within the range

$$\omega + vK \geq 0 \ , \tag{241}$$

or, equivalently, if

$$\omega(1 + \delta f_{1,2}) \geq 0 \ . \tag{242}$$

From (236) and (237) we have

$$1 + \delta f_1 = \frac{1}{2\delta}(1 + \sqrt{1 - 4\delta}) \ , \tag{243}$$

and

$$1 + \delta f_2 = \frac{1}{2\delta}(1 - \sqrt{1 - 4\delta}) \ . \tag{244}$$

Within the range $\cos(\alpha-w) > 0$, ω and δ have the same sign. Hence

$$1 + \delta f_1 > 0; \ \omega > 0 \quad ,$$

$$1 + \delta f_1 < 0; \ \omega < 0 \quad .$$

On the other hand,

$$1 + \delta f_2 > 0$$

for all ω. It then follows that for $\cos(\alpha-w) > 0$, (242) is satisfied for positive frequencies by f_1 and f_2, and for negative frequencies only by f_1. When $\cos(\alpha-w) < 0$, then

$$1 + \delta f_1 > 0; \ \omega < 0 \quad ,$$

$$1 + \delta f_1 < 0; \ \omega > 0 \quad ,$$

while $1 + \delta f_2 > 0$ for positive and negative ω. Consequently, for $\omega > 0$ a contribution arises only from f_2, while for $\omega < 0$, (242) has no solutions. These observations may be summarized as follows:

	$\omega>0$	$\omega<0$
$\cos(\alpha-w) > 0$	f_1, f_2	f_1
$\cos(\alpha-w) < 0$	f_2	NONE

Contributions from the second delta function, corresponding to Eq. (230b), are obtained if

$$\omega(1 + \delta f_{1,2}) < 0 \quad .$$

One then finds that the ω and $\cos(\alpha - w)$ regions bear the following relation to f_1, f_2:

	$\omega>0$	$\omega<0$
$\cos(\alpha-w) > 0$	NONE	f_2
$\cos(\alpha-w) < 0$	f_1	f_1, f_2

With the aid of the preceding results we can integrate Eq. (229) with respect to K. Since $\cos(\alpha - w) = 0$ defines the boundary in the wave number direction space on either side of which a different combination of the roots f_1 and f_2 contributes, it is convenient to change the integration variable from w to $\theta = w - \alpha$. After some algebraic manipulations one finds, for $\omega > 0$,

$$\Phi^{(V)}_{\nu\mu}(\omega,y) = \frac{\pi}{2}\frac{(\sigma\mu_o B_p)^2}{(1 + 3\cos^2\phi_D)} \cdot \omega$$

$$\left\{\left[\int_{-\frac{\pi}{2}}^{-\theta_m} + \int_{\theta_m}^{\frac{\pi}{2}}\right] d\theta\, g_{\nu\mu}(\theta+\alpha,\alpha)\sum_{\ell=1}^{2}\frac{\sqrt{f_\ell}\, e^{-2y\frac{\omega^2}{g}f_\ell}}{|2\delta\sqrt{f_\ell} - 1|}\,\psi\left(\frac{\omega^2}{g}f_\ell,\ \theta+\alpha\right)\right.$$

$$\left. + \left[\int_{\pi/2}^{\pi} + \int_{-\pi}^{-\pi/2}\right] d\theta\, g_{\nu\mu}(\theta+\alpha,\alpha)\sum_{\ell=1}^{2}\frac{\sqrt{f_\ell}\, e^{-2y\frac{\omega^2}{g}f_\ell}}{|2\delta\sqrt{f_\ell}-(-1)^{\ell}|}\,\psi\left[\frac{\omega^2}{g}f_\ell,\ \theta+\alpha+(2-\ell)\pi\right]\right\}, \quad (245)$$

where

$$\theta_m = \begin{cases} \cos^{-1}\frac{g}{4\omega V} & ; \ \frac{g}{4\omega V} < 1\,, \\ 0 & ; \ \frac{g}{4\omega V} > 1\,. \end{cases} \quad (246)$$

The quantities f_1, f_2 are functions of θ and ω, and are given by Eq. (236) and Eq. (237) with $\delta = \frac{\omega V \cos \theta}{g}$. The first integral comprises the two contributions f_1 and f_2 from the first delta function in Eq. (229); in the second integral, contribution f_2 arises from the first delta function, while contribution f_1 arises from the second delta function. The latter is associated with the "inverted" spectrum $\psi(-\underline{K})$, and the time domain dependence $\exp - i\Omega(K)\tau$. Thus, had we omitted this term in our original representation, i.e., used only the complex exponential representation for the correlation function of the form $\psi(\underline{K}) \exp i\Omega(K)\tau$, we would have obtained an incorrect result.

In the limiting case of zero platform velocity we find, from Eq. (239), that $f_1 \to \infty$ while $f_2 \to 1$. If the wave number spectrum $\psi(\underline{K})$ in Eq. (245) decays with K, then only the terms involving f_2 give nonzero contributions. The two integrands in Eq. (245) then become identical. Since $\theta_m \to 0$ as $V \to 0$, one can write Eq. (245) as a single integral between limits of $-\pi$ and π. The whole expression then reduces to the spectrum in the stationary frame of reference, given in Eq. (204).

The expression for the gradient spectra follows from Eq. (245) through a replacement of $g_{\nu\mu}$ by $2g_{\nu\mu;rs}$ and multiplication of each term of the series by $K_\ell^2 = \frac{\omega^4}{g^2} f_\ell^2$:

$$\Phi^{(V)}_{\nu\mu;rs}(\omega,y) = \frac{\pi (\sigma\mu_o B_p)^2}{g^2(1 + 3\cos^2 \phi_D)} \quad \omega^5$$

$$\left\{ \left[\int_{-\frac{\pi}{2}}^{-\theta_m} + \int_{\theta_m}^{\pi/2} \right] d\theta \, g_{\nu\mu;rs}(\theta+\alpha,\alpha) \sum_{\ell=1}^{2} \frac{f_\ell^{5/2} e^{-2y \frac{\omega^2}{g} f_\ell}}{|2\delta \sqrt{f_\ell} - 1|} \, \psi\!\left(\frac{\omega^2}{g} f_\ell, \theta+\alpha\right) \right.$$

$$\left. + \left[\int_{\pi/2}^{\pi} + \int_{-\pi}^{-\pi/2} \right] d\theta \, g_{\nu\mu;rs}(\theta+\alpha,\alpha) \sum_{\ell=1}^{2} \frac{f_\ell^{5/2} e^{-2y \frac{\omega^2}{g} f_\ell}}{|2\delta \sqrt{f_\ell} - (-1)^\ell|} \, \psi\left[\frac{\omega^2}{g} f_\ell, \theta+\alpha+(2-\ell)\pi\right] \right\} . \tag{247}$$

The interpretation of the various contributing terms in Eqs. (247) and (245) is facilitated if we assume a unidirectional form of the surface wave spectrum. For definiteness, assume that the dependence of $\psi(\underline{K})$ on K is identical to that in the Pierson-Neumann spectrum, viz.,

$$\psi(\underline{K}) = \frac{1}{4} \overline{C} g^2 \frac{\pi}{2} (Kg)^{-5/2} \exp\left\{- 2g/KU^2\right\} \delta(w - w_o) \ .$$

The additional factor of $\pi/2$ has been included to ensure the same normalization over the complete range of w as implied by the $\cos^2(w - w_o)$ directional dependence. We then find that the magnetic field component spectrum in Eq. (245) becomes

$$\Phi_{\nu\mu}^{(V)}(\omega,y) = \frac{\pi^2 g^2 \overline{C}(\sigma\mu_o B_p)^2}{16 (1 + 3 \cos^2 \phi_D)} \omega^{-8}$$

$$\cdot \begin{cases} g_{\nu\mu}(w_o,\alpha) \dfrac{\exp - \left\{2y \frac{\omega^2}{g} f_2 + 2g^2\omega^{-2}U^{-2}f_2^{-1}\right\}}{|2\delta \sqrt{f_2} - 1| \, f_2^{*}} \ ; & |w_o - \alpha| > \frac{\pi}{2} \ , \\ g_{\nu\mu}(w_o,\alpha) \, U\left[\dfrac{g}{4V \cos(w_o - \alpha)} - \omega\right] \displaystyle\sum_{\ell=1}^{2} \dfrac{\exp - \left\{2y \frac{\omega^2}{g} f_\ell + 2g^2\omega^{-2}U^{-2}f_\ell^{-1}\right\}}{|2\delta \sqrt{f_\ell} - 1| \, f_\ell^{*}} & \\ + g_{\nu\mu}(w_o - \pi,\alpha) \dfrac{\exp - \left\{2y \frac{\omega^2}{g} f_1 + 2g^2\omega^{-2}U^{-2}f_1^{-1}\right\}}{|2\delta \sqrt{f_1} + 1| \, f_1^{*}} \ ; & |w_o - \alpha| < \frac{\pi}{2} \ , \end{cases} \tag{248}$$

where $U(x)$ is the unit step function, and δ and the f_ℓ are to be evaluated at the same value of w (viz., w_o or $w_o - \pi$) as the corresponding projection factor $g_{\nu\mu}(w)$. The proper combination of contributing terms in the spectrum evidently depends on the relative orientation of the platform velocity and the direction

of propagation of the surface wave. Thus, if $|w_o - \alpha|$ is greater than 90°, only the first term in Eq. (248) contributes. Within the two forward quadrants, $|w_o - \alpha| < \pi/2$, there are two contributing terms, the first of which vanishes for $V \cos(w_o - \alpha) > g/4\omega$, i.e., when the projection of the platform velocity vector on the surface wave propagation vector is greater than one half the group velocity of the surface wave (recall that $v_g = d\omega/dK = g/2\omega$). The last term describes a pure motion-induced effect, in that it tends to zero as $V \to 0$. This follows from the fact that as $V \to 0$, $f_1 \to \infty$. When the platform trajectory is orthogonal to the surface wave motion, i.e., $w_o - \alpha = \pm \pi/2$, then $f_1 \to \infty$, while $f_2 \to 1$. Thus, only the two terms comprising $f_2 = 1$ contribute, and the spectrum is the same as obtains in a stationary reference frame.

When the surface wave spectrum is not purely unidirectional, Eqs. (245) and (247) have to be evaluated numerically. A limited number of such calculations has been carried out for the Pierson-Neumann spectrum with a $\cos^2(w - w_o)$ directional dependence. The results are shown in Figs. 13 and 14 for platform motions of 100 m/sec (typical aircraft speeds). Figure 13 shows spectra of the horizontal magnetic field component (measured along the track) for three values of wind speed. Figure 13a gives the spectra at 50m and Fig. 13b at 100m above the ocean surface. In all cases $\alpha = 0$ (aircraft flying in the plane containing the geomagnetic field, see Fig. 3), $\phi_D = 0$ (equatorial zone), and $w_o = 0$ (wind direction along the track). Figure 14 shows the results for the HV gradient, i.e., $\Phi_{12;12}$. Again $\phi_D = 0$, $\alpha = 0$, and $w_o = 0$. If we take the sensitivity of an "average" superconducting gradiometer at $.1(\mathrm{pT/m})^2/\mathrm{Hz}$, then a magnetic field gradient induced by 10 m/sec wind waves is barely detectable at 50m above the ocean surface, and not detectable at 100 m.

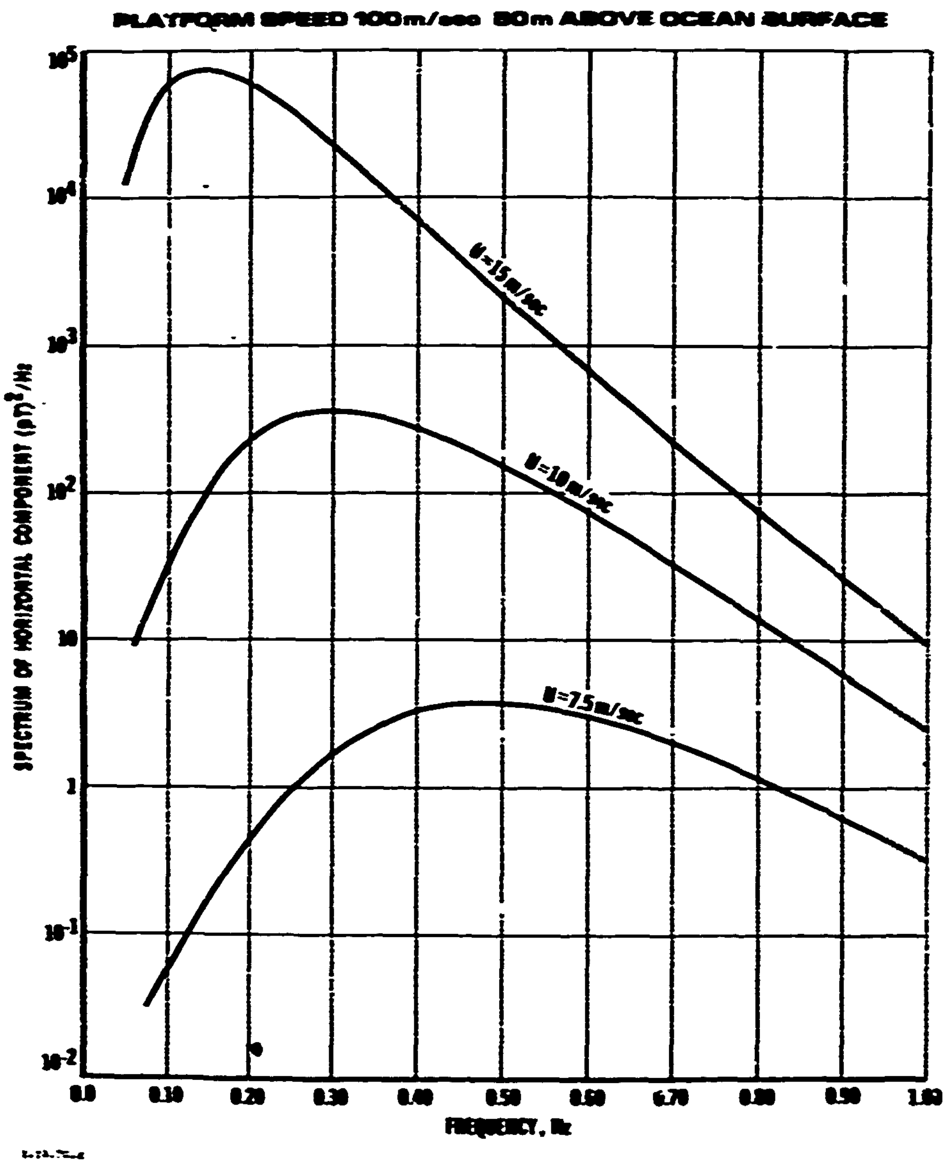

FIGURE 13a. Spectrum of horizontal component of magnetic field induced by linear surface waves (ϕ_D = 0, α = 0, component along the track of platform motion).

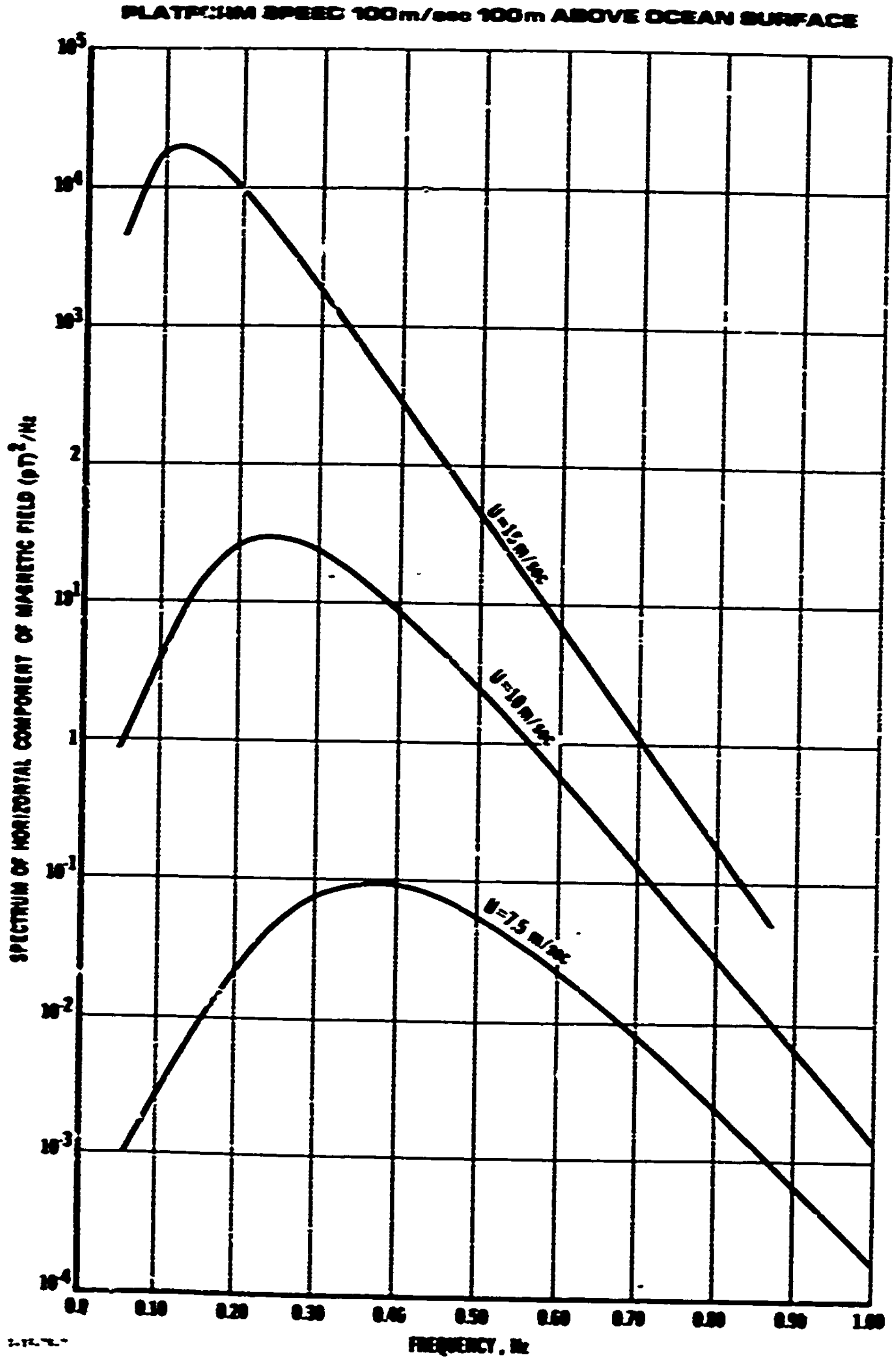

FIGURE 13b. Spectrum of horizontal component of magnetic field induced by linear surface waves ($\phi_D = 0$, $\alpha = 0$, component along the track of platform motion).

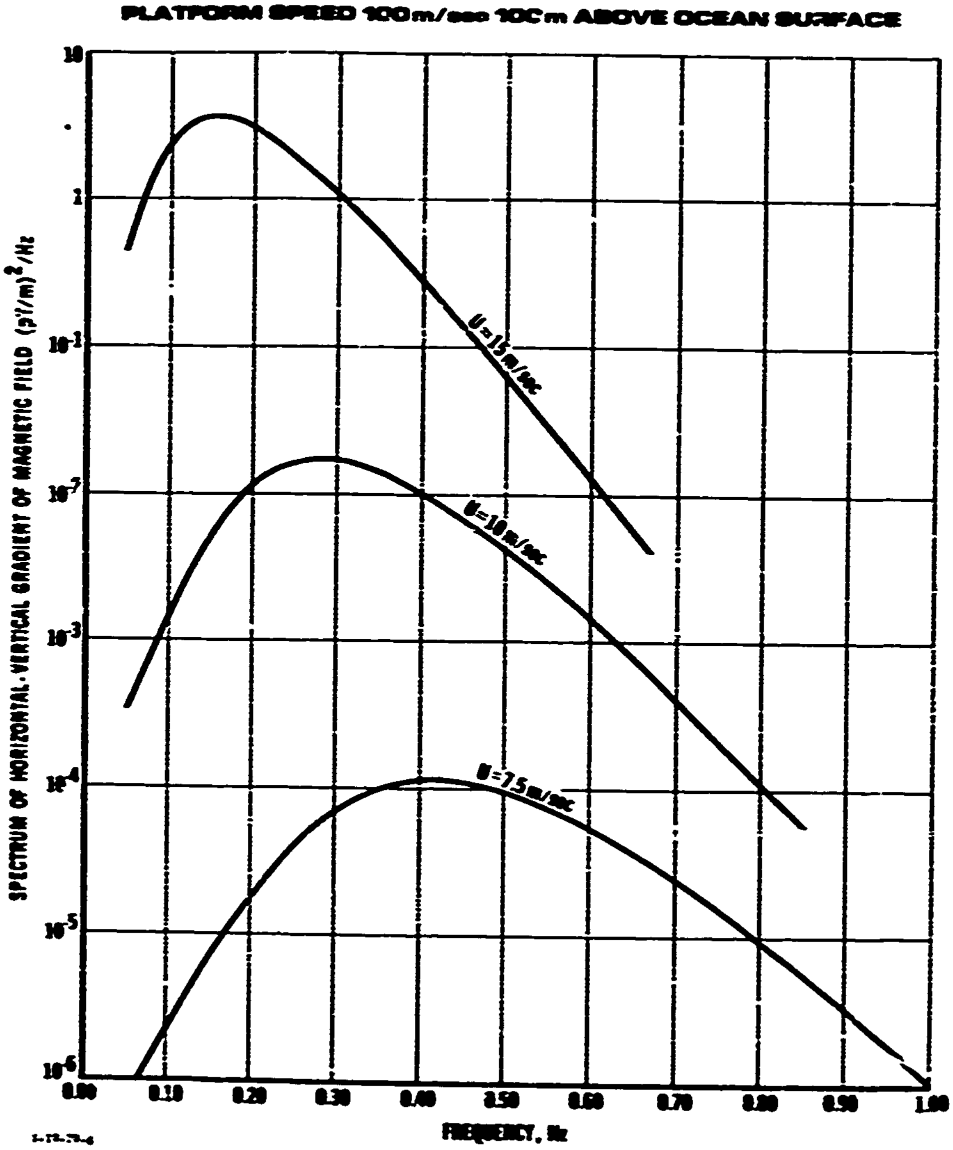

FIGURE 14a. Spectrum of horizontal-vertical gradient of magnetic field induced by linear surface waves ($\phi_D = 0$, $\alpha = 0$).

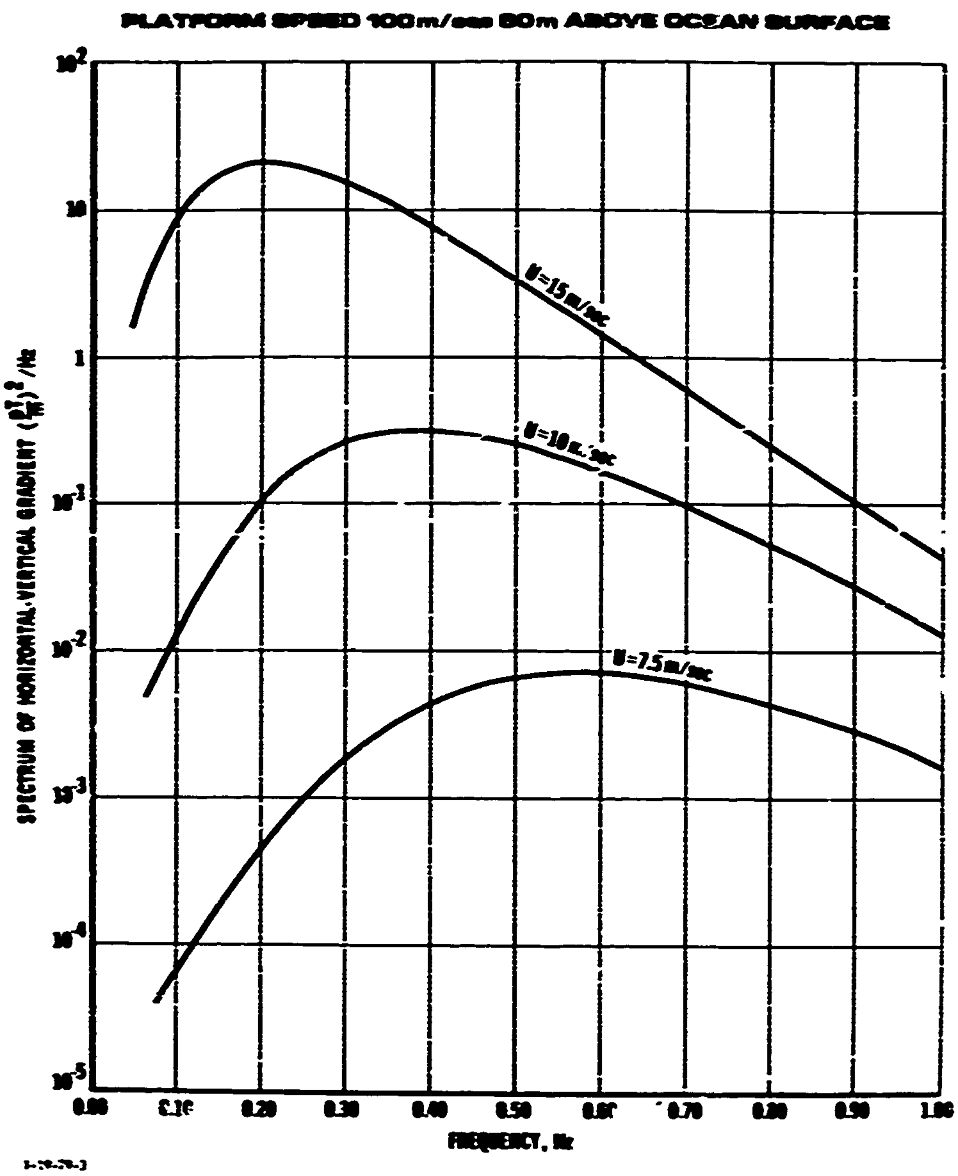

FIGURE 14b. Spectrum of horizontal-vertical gradient of magnetic field induced by linear surface waves ($\phi_D = 0$, $\alpha = 0$).

B. INTERNAL-WAVE-INDUCED MAGNETIC FIELD SPECTRA OBSERVED FROM A MOVING PLATFORM

While the formal procedure for finding spectra of internal-wave-induced magnetic fields and gradients as modified by platform motion is the same as just described for surface waves, the details are somewhat different. Thus, one can no longer obtain an explicit solution for the roots of the augmented dispersion relations. Moreover, a summation is required over an infinite number of modes.

We shall not start with the general relations Eqs. (151) and (180), but with the special form Eq. (157) together with the corresponding expression for gradients. We then have the following representation for the component spectra:

$$\Phi^{(V)}_{\nu\mu}(\omega,y) = \frac{(\sigma\mu_o B_p)^2}{1+3\cos^2\phi_D} \int_0^{2\pi} dw\, g_{\nu\mu}(w,\alpha) \int_0^{\infty} \frac{dK}{K^2} e^{-2Ky}$$

$$\cdot \sum_n L_n^2(K)\Omega_n^4(K) \left\{ I(K,w)\delta[\Omega_n - \underline{K}\cdot\underline{V} - \omega] + I(K,w+\pi)\delta[\Omega_n + \underline{K}\cdot\underline{V} + \omega] \right\} . \qquad (249)$$

It will be convenient to change the variable integration from w to $\beta = w - \alpha + \pi$. Equation (249) may then be recast to read

$$\Phi^{(V)}_{\nu\mu}(\omega,y) = \frac{(\sigma\mu_c B_p)^2}{1+3\cos^2\phi_D} \cdot \int_0^{\pi} d\beta \int_0^{\infty} \frac{dK}{K^2} \epsilon^{-2Ky}$$

$$\cdot \sum_n L_n^2(K)\Omega_n^4(K) \left\{ \xi_{\nu\mu}(K,\alpha,\beta)\delta[\Omega_n + KV\cos\beta - \omega] + \eta_{\nu\mu}(K,\alpha,\beta)\delta[\Omega_n - KV\cos\beta + \omega] \right\}, \qquad (250)$$

where

$$\xi_{\nu\mu}(K,\alpha,\beta) = g_{\nu\mu}(\alpha+\beta-\pi,\alpha)I(K,\alpha+\beta-\pi) + g_{\nu\mu}(\alpha-\beta-\pi,\alpha)I(K,\alpha-\beta-\pi) , \tag{251a}$$

$$\eta_{\nu\mu}(K,\alpha,\beta) = g_{\nu\mu}(\alpha+\beta-\pi,\alpha)I(K,\alpha+\beta) + g_{\nu\mu}(\alpha-\beta-\pi,\alpha)I(K,\alpha-\beta) . \tag{251b}$$

Equation (250) is of the same general form as Eq. (E-138) in Appendix E. Consequently, the result of integrating Eq. (250) with respect to K can be obtained directly from the expressions for the towed spectrum of the vertical velocity of internal waves, as given by Eqs. (E-154) and (E-155). Referring to Eq. (E-154), and replacing $j(K,\alpha+\pi,\beta)$ and $j(K,\alpha,\beta)$ with $\xi_{\nu\mu}(K,\alpha,\beta)$ and $\eta_{\nu\mu}(K,\alpha,\beta)$, respectively, and finally multiplying the resulting expressions by the constant factor in front of Eq. (250), yields

$$\Phi_{\nu\mu}^{(V)}(\omega,y) = \frac{(\sigma\mu_o B_p)^2}{1+3\cos^2\phi_D} \cdot$$

$$\int_0^{\pi/2} d\beta \left\{ \sum_n \frac{\xi_{\nu\mu}(\kappa_n,\alpha,\beta)L_n^2(\kappa_n)\Omega_n^4(\kappa_n)\, e^{-2\kappa_n y}}{[v_{gn}(\kappa_n)+V\cos\beta][\kappa_n]^2} + \frac{\eta_{\nu\mu}(\kappa_n^{(3)},\alpha,\beta)L_n^2(\kappa_n^{(3)})\Omega_n^4(\kappa_n^{(3)})\, e^{-2\kappa_n^{(3)}y}}{[V\cos\beta-v_{gn}(\kappa_n^{(3)})][\kappa_n^{(3)}]^2} \right\}, \tag{251}$$

which is valid for $\omega > N_{max}$ (N_{max} the maximum Väisälä frequency). For $\omega < N_{max}$ we employ Eq. (E-155) to obtain

$$\Phi_{\nu\mu}^{(V)}(\omega,y) = \frac{(\sigma\mu_o B_p)^2}{1+3\cos^2\phi_D} \cdot$$

$$\left\{ \int_0^{\pi/2} d\beta \sum_n \left[\frac{\xi_{\nu\mu}(\kappa_n,\alpha,\beta)L_n^2(\kappa_n)\Omega_n^4(\kappa_n)\, e^{-2\kappa_n y}}{[v_{gn}(\kappa_n)+V\cos\beta][\kappa_n]^2} + \frac{\eta_{\nu\mu}(\kappa_n^{(3)},\alpha,\beta)L_n^2(\kappa_n^{(3)})\Omega_n^4(\kappa_n^{(3)})\, e^{-2\kappa_n^{(3)}y}}{[V\cos\beta-v_{gn}(\kappa_n^3)][\kappa_n^{(3)}]^2} \right] \right.$$

$$\left. + \sum_{\ell=1}^{2} \sum_n \int_{\pi/2}^{\beta_{max}^{(n)}} d\beta \, \frac{\xi_{\nu\mu}(\kappa_n^{(\ell)},\alpha,\beta)L_n^2(\kappa_n^{(\ell)})\Omega_n^4(\kappa_n^{(\ell)})\, e^{-2\kappa_n^{(\ell)}y}}{|v_{gn}(\kappa_n^{(\ell)})+V\cos\beta|[\kappa_n^{(\ell)}]^2} \right\} . \tag{252}$$

The four classes of roots, κ_n, $\kappa_n^{(1)}$, $\kappa_n^{(2)}$, $\kappa_n^{(3)}$, entering into these expressions are defined in Eqs. (E-151), (E-152), and (E-153), and a graphical construction to estimate their location in the ωK space is shown in Fig. E-3.

An exact numerical evaluation of Eq. (251) and Eq. (252) would be rather difficult. Fortunately, for the case of greatest interest, viz., for platform velocities much larger than the maximum internal wave group speed *and* $\omega > N_{max}$, the spectrum can be approximated by a fairly simple asymptotic expression. This asymptotic approximation is discussed in Appendix E. Based on the same justification as presented in the discussion following Eq. (E-155), the asymptotic form of Eq. (251) becomes

$$\Phi_{\nu\mu}^{(V)}(\omega,y) \sim \frac{(\sigma\mu_o B_p)^2}{1 + 3\cos^2\phi_D}$$

$$\cdot \frac{1}{\omega}\int_0^{\pi/2} d\beta \left[\xi_{\nu\mu}(K,\alpha,\beta) + \eta_{\nu\mu}(K,\alpha,\beta)\right] \cdot \frac{e^{-2Ky}}{K}\sum_n L_n^2(K)\,\Omega_n^4(K) \; , \qquad (253)$$

where

$$K = \omega(V\cos\beta)^{-1} \; .$$

The closed form of the last sum is given in Eq. (160), or, in the case of a deep ocean, by Eq. (161). Substituting the latter in Eq. (253) yields

$$\Phi_{\nu\mu}^{(V)}(\omega,y) \sim \frac{(\sigma\mu_o B_p)^2}{4(1 + 3\cos^2\phi_D)}$$

$$\cdot \frac{1}{\omega}\int_0^{\pi/2} d\beta\left[\xi_{\nu\mu}(K,\alpha,\beta) + \eta_{\nu\mu}(K,\alpha,\beta)\right] K\, e^{-2Ky}\int_{-\infty}^{0} y'^2 N^2(y')\, e^{2Ky'}\, dy' \; . \qquad (254)$$

Suppose we now assume an isotropic excitation function of the form $I(\underline{K}) = CK^{-p}$. Then Eq. (254) becomes

$$\Phi_{\nu\mu}^{(V)}(\omega,y) \sim \frac{(\sigma\mu_o B_p)^2\, c}{2(1 + 3\cos^2\phi_D)\omega} \int_0^{\pi/2} d\beta\, \tilde{g}_{\nu\mu}(\alpha,\beta)\, K^{-p+1}\, e^{-2Ky} \int_{-\infty}^{0} y'^2 N^2(y')\, e^{2Ky'}\, dy' , \tag{255}$$

where

$$\tilde{g}_{\nu\mu}(\alpha,\beta) = g_{\nu\mu}(\alpha+\beta-\pi,\alpha) + g_{\nu\mu}(\alpha-\beta-\pi,\alpha) \quad . \tag{256}$$

The expressions for the spectra of magnetic field gradients are obtained from the component spectra by the replacement of the projection factors $g_{\nu\mu}$ by $2g_{\nu\mu;rs}$ and multiplication of the integrands by the square of the wave number. Thus, the general expressions for gradient spectra $\Phi_{\nu\mu;rs}^{(V)}(\omega,y)$ follow from Eq. (251) and (252) by deleting the factors $[\kappa_n]^2$, $[\kappa_n^{(s)}]^2$, $[\kappa_n^{(\ell)}]^2$ appearing in the denominators, and replacing $\xi_{\nu\mu}$, $\eta_{\nu\mu}$ by $2\xi_{\nu\mu;rs}$ and $2\eta_{\nu\mu;rs}$, respectively. The latter are defined by

$$\xi_{\nu\mu;rs}(K,\alpha,\beta) = g_{\nu\mu;rs}(\alpha+\beta-\pi,\alpha)\, I(K,\alpha+\beta-\pi) + g_{\nu\mu;rs}(\alpha-\beta-\pi,\alpha)\, I(K,\alpha-\beta-\pi) \quad , \tag{257a}$$

$$\eta_{\nu\mu;rs}(K,\alpha,\beta) = g_{\nu\mu;rs}(\alpha+\beta-\pi,\alpha)\, I(K,\alpha+\beta) + g_{\nu\mu;rs}(\alpha-\beta-\pi,\alpha)\, I(K,\alpha-\beta) \quad . \tag{257b}$$

In particular, the asymptotic form of the magnetic field gradient spectrum is

$$\Phi_{\nu\mu;rs}^{(V)}(\omega,y) \sim \frac{(\sigma\mu_o B_p)^2}{2(1 + 3\cos^2\phi_D)} \cdot \frac{1}{\omega} \int_0^{\pi/2} d\beta \left[\xi_{\nu\mu;rs}(K,\alpha,\beta) + \eta_{\nu\mu;rs}(K,\alpha,\beta)\right] K^3\, e^{-2Ky} \int_{-\infty}^{0} y'^2 N^2(y')\, e^{2Ky'}\, dy' . \tag{258}$$

In the special case of an isotropic internal wave excitation function with power law dependence on wave number, this reduces to

$$\Phi^{(V)}_{\nu\mu;rs}(\omega,y) \sim \frac{C(\sigma\mu_0 B_p)^2}{(1+3\cos^2\phi_D)\,\omega} \int_0^{\pi/2} d\beta\, \tilde{g}_{\nu\mu;rs}(\alpha,\beta)\, K^{-p+3}\, e^{-2Ky} \int_{-\infty}^{0} y'^2 N^2(y')\, e^{2Ky'}\, dy' \, , \tag{259}$$

where

$$\tilde{g}_{\nu\mu;rs}(\alpha,\beta) = g_{\nu\mu;rs}(\alpha+\beta-\pi,\alpha) + g_{\nu\mu;rs}(\alpha-\beta-\pi,\alpha) \quad . \tag{260}$$

Since typically the horizontal group velocity of internal waves is on the order of only a fraction of a meter/sec, the asymptotic expressions for the spectra, Eqs. (254) (255) (258) and (259), are valid at fairly moderate platform velocities. It is important to note, however, that these expressions cease to be valid in the frequency range $\omega < N_{max}$, even for fast platform velocities.

Equations (255) and (259) were evaluated for an exponentially stratified ocean with the same parameters as employed in Chapter VI. The results are plotted in Figs. 15 and 16. Figures 15a,b,c show the spectra of the horizontal component of the magnetic field along the track of the platform motion. For comparison, the spectra of the surface-wave-induced horizontal magnetic field component along the track are included. In all cases $\phi_D = 0$, $\alpha = 0$. The internal-wave-induced magnetic field component spectra were computed for the two extreme values of p.

Figures 16a,b show plots of the horizontal-vertical gradient spectrum, again for $\alpha = 0$, $\phi_D = 0$. If we take the gradiometer sensitivity equal to $.1(pT)^2/m^2Hz$, then at a height of 50m, detectable internal wave levels appear to be attained only at frequencies below about .05 Hz.

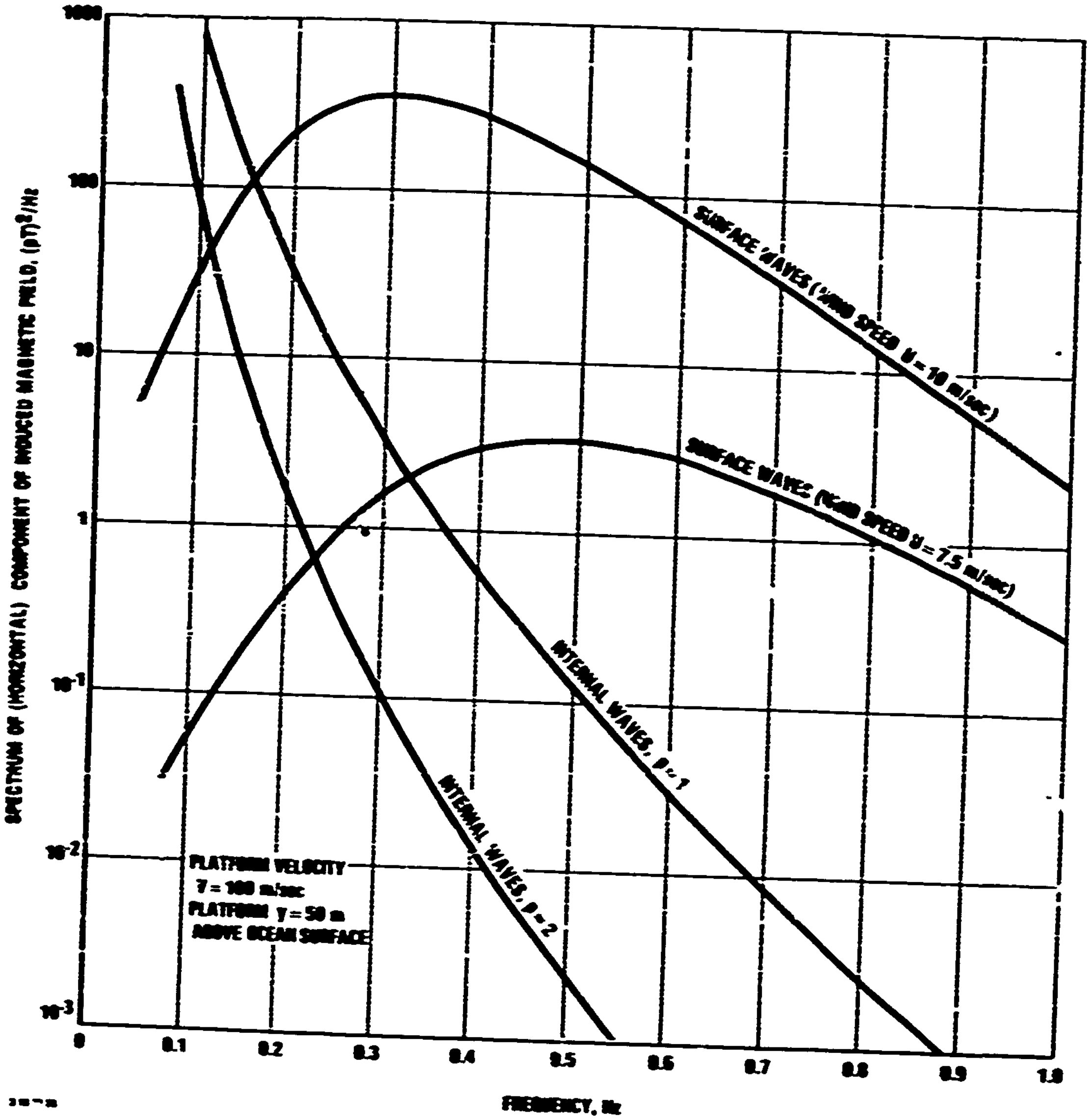

FIGURE 15a. Spectrum of (horizontal) component of induced magnetic field ($\alpha = 0$, $\phi_D = 0$)

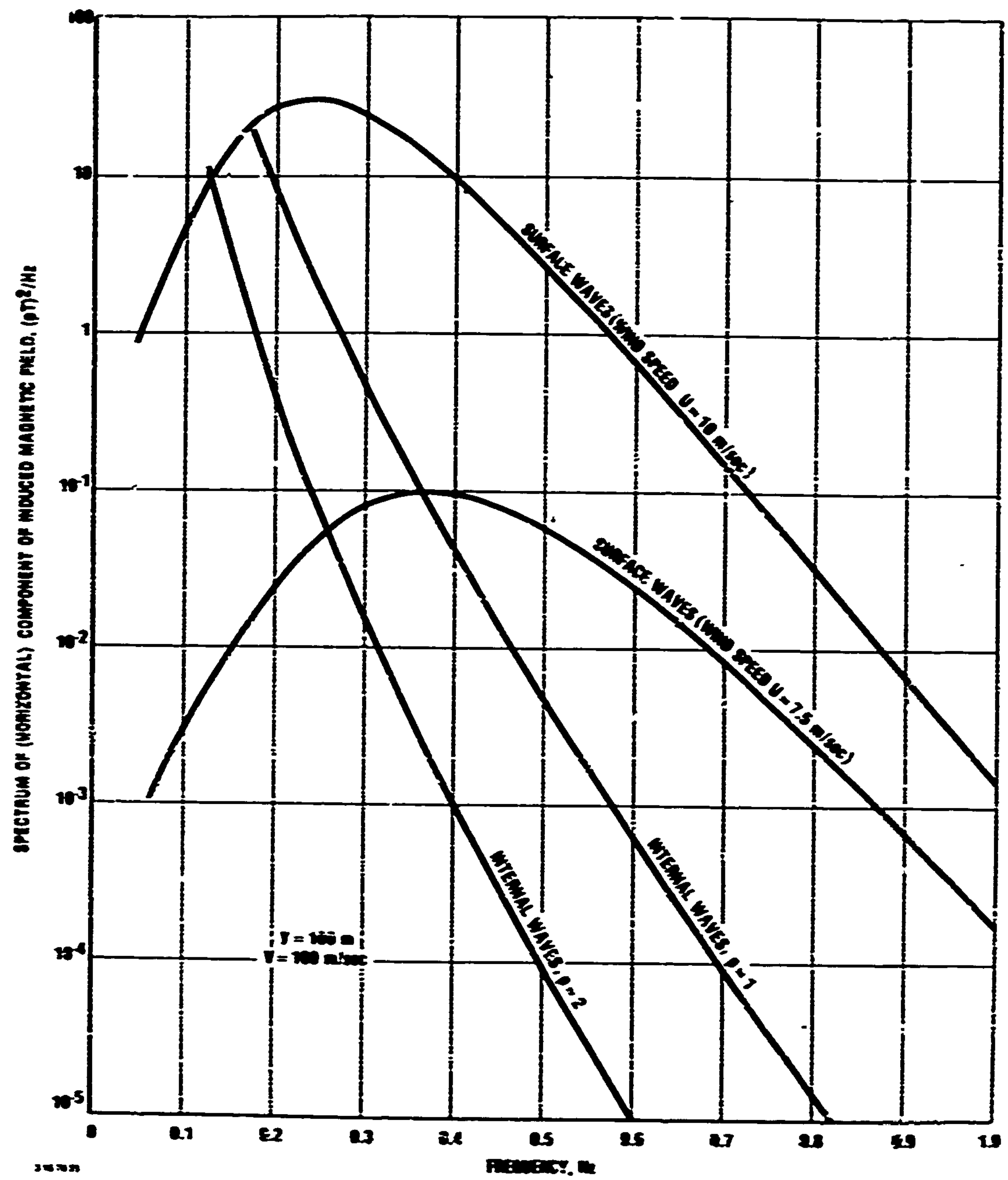

FIGURE 15b. Spectrum of (horizontal) component of induced magnetic field ($\alpha = 0$, $\phi_D = 0$)

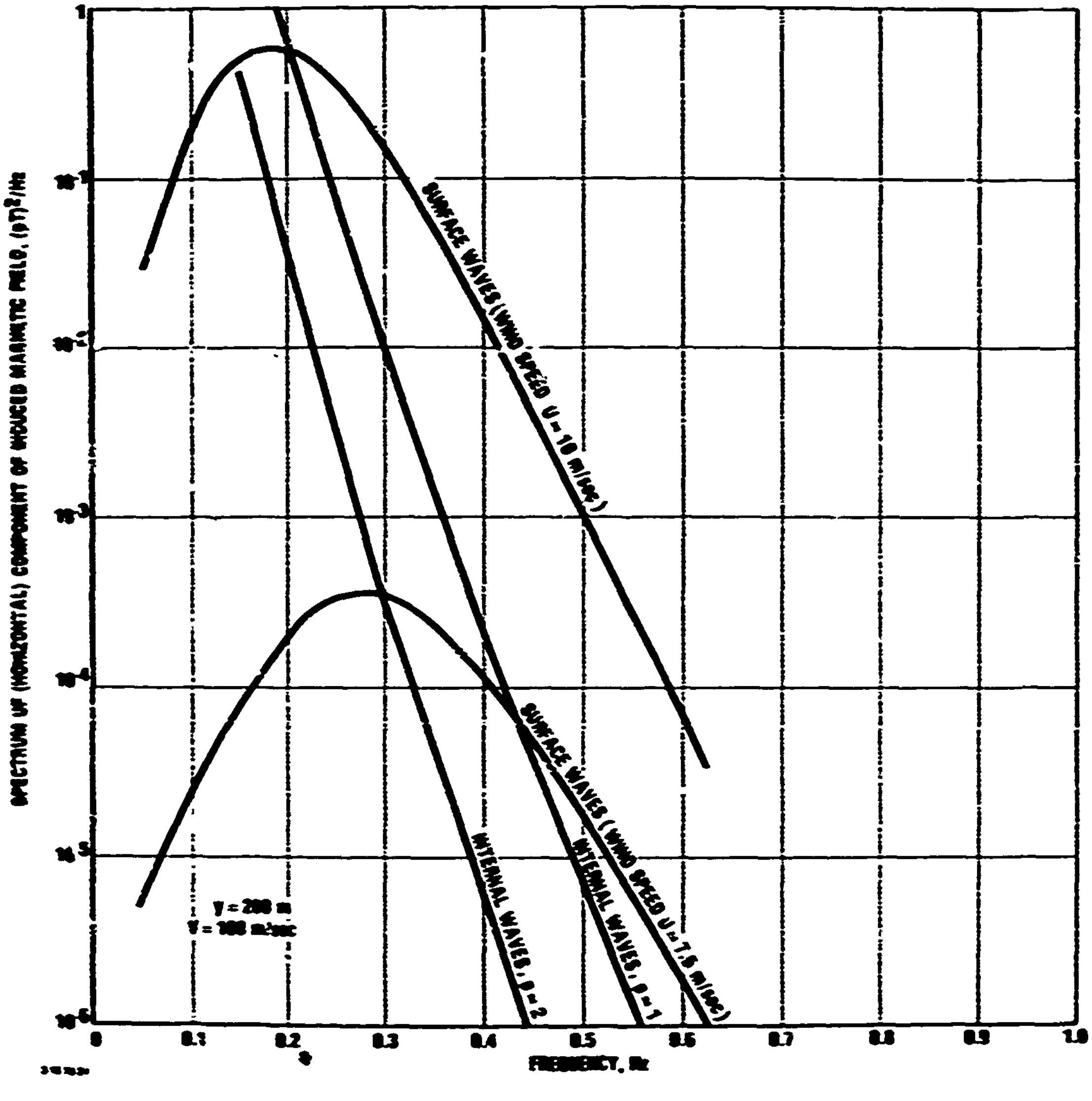

FIGURE 15c. Spectrum of (horizontal) component of induced magnetic field (α = 0, ϕ_D = 0)

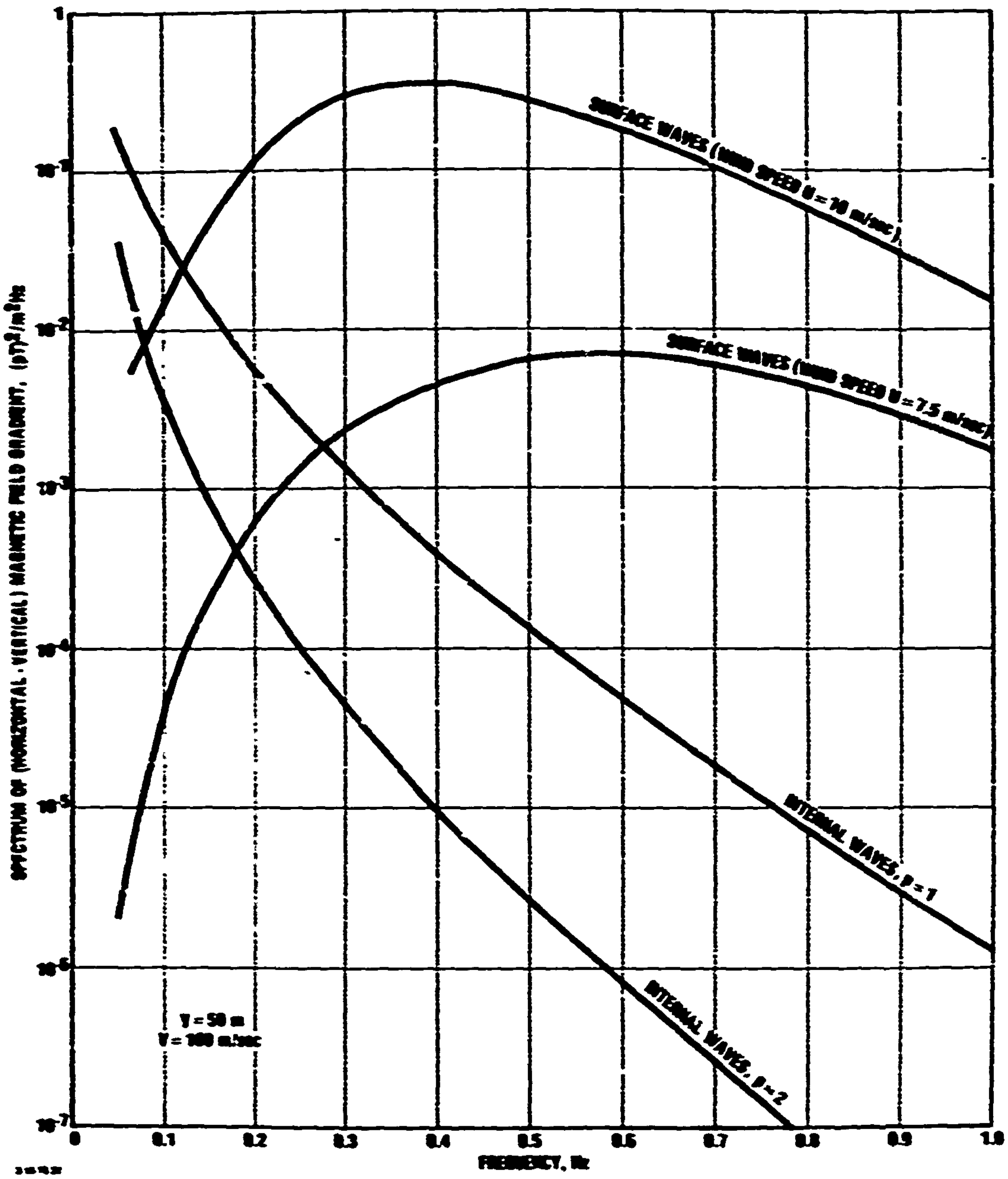

FIGURE 16a. Spectrum of (horizontal-vertical) magnetic field gradient ($\alpha = 0$, $\phi_D = 0$)

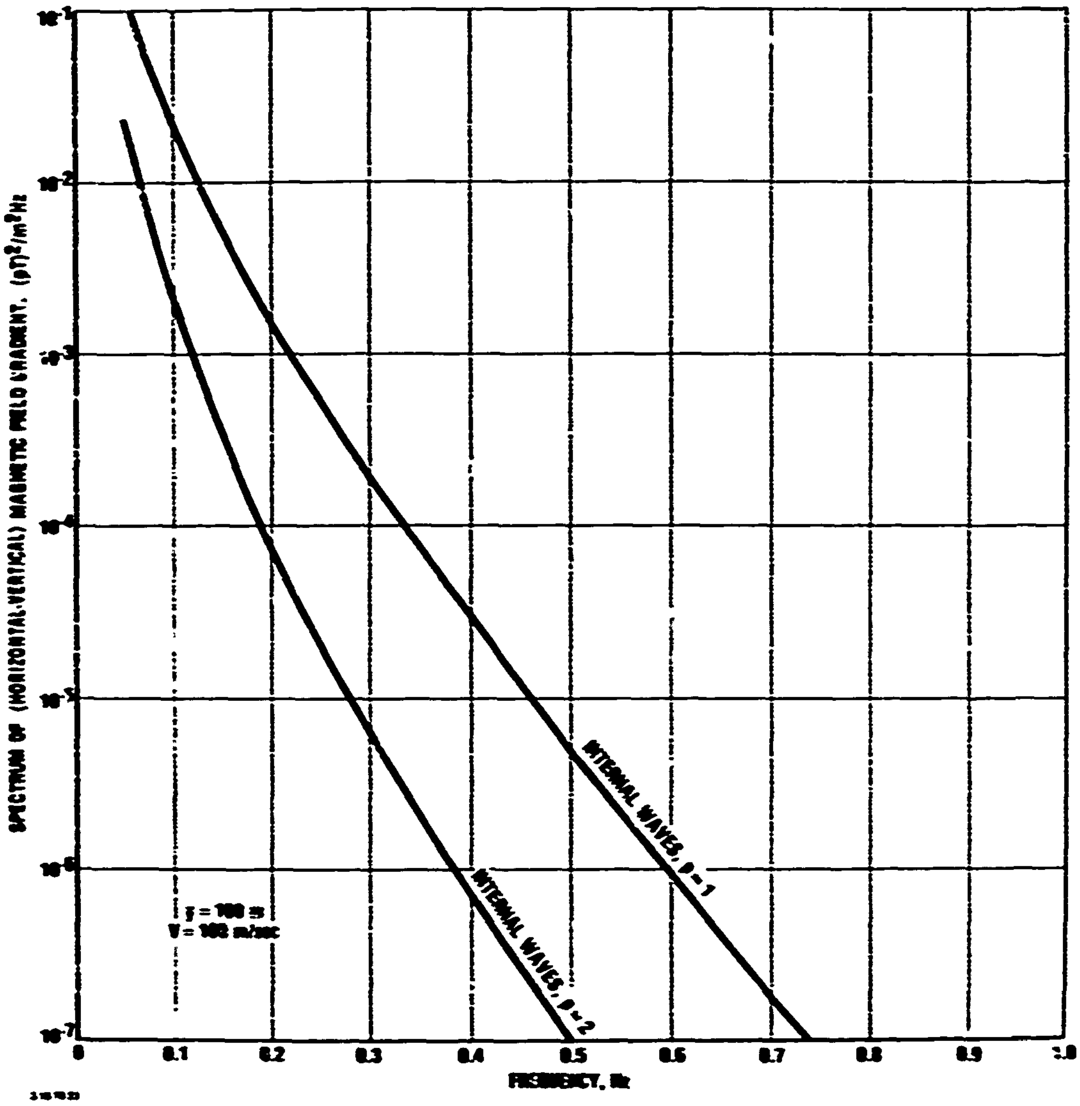

FIGURE 16b. Spectrum of (horizontal-vertical) magnetic field gradient (α = 0, ϕ_D = 0)

APPENDIX A

SMALL-AMPLITUDE OCEAN SURFACE WAVES

APPENDIX A

SMALL-AMPLITUDE OCEAN SURFACE WAVES

Here we present a concise account of the theory of small-amplitude surface waves that is relevant to the computation of induced magnetic fields. For a more detailed treatment the reader could refer to [11] and [12].

Quite generally, the vertical displacement $\eta(\rho,t)$ of the ocean surface may be represented by the Fourier integral

$$\eta(\underline{\rho},t) = \iint_{-\infty}^{\infty} e^{-i\underline{k}_T \cdot \underline{\rho}} F(\underline{k}_T,t)\, d^2\underline{k}_T \quad . \tag{A-1}$$

In the coordinate system adapted herein, y is the local vertical, the mean vertical displacement of the ocean surface is coincident with the xz plane, and $y \geq 0$ defines the region above the ocean. The function $F(\underline{k}_T,t)$, just as $\eta(\rho,t)$, provides a kinematic description of the ocean surface. When the surface displacement is modeled as a spatially homogeneous stochastic process with zero mean, there can be no correlation between $F(\underline{k}_T',t)$ and $F(\underline{k}_T'',t)$ unless $\underline{k}_T' = \underline{k}_T''$. Formally*, this fact may be expressed as follows:

$$\langle F(\underline{k}_T',t_1)\, F^*(\underline{k}_T'',t_2)\rangle = S_{\eta\eta}(\underline{k}_T',t_1,t_2)\, \delta(\underline{k}_T' - \underline{k}_T'') \tag{A-2}$$

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*These results can, of course, also be phrased rigorously in terms of the Stieltjes-Lebesque integral (see, e.g., [13]). Here we avoid such mathematical refinements.

With the correlation function of the displacement denoted by $R_{\eta\eta}$ one has

$$R_{\eta\eta}(\underline{\rho}_1-\underline{\rho}_2,t_1,t_2) = \langle \eta(\underline{\rho}_1,t_1)\ \eta(\underline{\rho}_2,t_2)\rangle \quad , \tag{A-3}$$

Thus, (A-2) in conjunction with (A-1) yields

$$R_{\eta\eta}(\underline{\rho},t_1,t_2) = \iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}}\, S_{\eta\eta}(\underline{k}_T,t_1,t_2)\, d^2\underline{k}_T \quad . \tag{A-4}$$

The quantity $S_{\eta\eta}(\underline{k}_T,t_1,t_2)$ is the spatial cross-spectrum of the water displacement. In general, it will be a function of the time reference points t_1 and t_2. If we also suppose that the stochastic process $\eta(\underline{\rho},t)$ is stationary in time, then $S_{\eta\eta}$ is a function of $t_1-t_2 = \tau$. We then have

$$R_{\eta\eta}(\underline{\rho},\tau) = \iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}}\, S_{\eta\eta}(\underline{k}_T,\tau)\, d^2\underline{k}_T \quad . \tag{A-5}$$

The spatial cross-spectrum $S_{\eta\eta}(\underline{k}_T,\tau)$ has several symmetry properties. Thus, from the definition (A-3), we have

$$R_{\eta\eta}(-\underline{\rho},-\tau) = R_{\eta\eta}(\underline{\rho},\tau) \quad . \tag{A-6}$$

By virtue of (A-5), $S_{\eta\eta}(\underline{k}_T,\tau)$ must possess the same symmetry property in $\underline{k}_T,\tau$, viz.,

$$S_{\eta\eta}(\underline{k}_T,\tau) = S_{\eta\eta}(-\underline{k}_T,-\tau) \quad . \tag{A-7}$$

In particular, $S_{\eta\eta}(\underline{k}_T,0) = S_{\eta\eta}(-\underline{k}_T,0)$. Also, since $R_{\eta\eta}(\underline{\rho},\tau)$ is real, (A-5) requires that

$$S_{\eta\eta}(\underline{k}_T,\tau) = S^*_{\eta\eta}(-\underline{k}_T,\tau) \quad . \tag{A-8}$$

Clearly, (A-7) and (A-8) together imply

$$S_{\eta\eta}(\underline{k}_T,\tau) = S^*_{\eta\eta}(\underline{k}_T,-\tau) \quad . \tag{A-9}$$

Thus, $S_{\eta\eta}(0,\tau)$ and $S_{\eta\eta}(\underline{k}_T,0)$ are real functions. The latter will be referred to as the spatial spectrum. When integrated over the wave number space $\underline{k}_T$, it gives the mean of the squared deviation of the ocean surface at any point $\underline{\rho}$:

$$\langle \eta^2(\underline{\rho},t)\rangle = R_{\eta\eta}(0,0) = \iint_{-\infty}^{\infty} S_{\eta\eta}(\underline{k}_T,0)\, d^2\underline{k}_T \quad . \tag{A-10}$$

It may also be shown [13] that $S_{\eta\eta}(\underline{k}_T,0) \geq 0$. Another quantity of interest is the temporal cross-spectrum $\Phi_{\eta\eta}(\underline{\rho},\omega)$, defined by

$$\Phi_{\eta\eta}(\underline{\rho},\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} R_{\eta\eta}(\underline{\rho},\tau)\, d\tau \quad . \tag{A-11}$$

By repeating the reasoning leading to (A-8) and (A-9) one finds that $\Phi_{\eta\eta}(\underline{\rho},\omega)$ obeys the symmetry relations

$$\Phi_{\eta\eta}(\underline{\rho},\omega) = \Phi^*_{\eta\eta}(\underline{\rho},-\omega) \quad . \tag{A-12}$$

$$\Phi_{\eta\eta}(\underline{\rho},\omega) = \Phi^*_{\eta\eta}(-\underline{\rho},\omega) \quad , \tag{A-13}$$

In particular, $\Phi_{\eta\eta}(\underline{\rho},0)$ and $\Phi_{\eta\eta}(0,\omega)$ are real functions and $\Phi_{\eta\eta}(0,\omega) \geq 0$. The latter quantity will be referred to as the temporal spectrum of $\eta(\underline{\rho},t)$. Its integral over frequency yields the statistical mean of the squared displacement $\eta(\underline{\rho},t)$, viz.,

$$\langle \eta^2(\underline{\rho},t)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(0,\omega)\, d\omega \quad . \tag{A-14}$$

Thus, for a stationary and spatially homogeneous stochastic process, the mean of the squared displacement $\langle\eta^2\rangle$ may be computed either from a knowledge of the spatial or the temporal spectrum, i.e., via formula (A-10) or (A-14).

The preceding relations are *purely kinematic* in that they deal only with the description of the ocean surface displacement per se, without an explicit reference to the velocity fields associated with such a displacement. Usually, it is the surface displacement alone that is subject to direct measurements. Thus, an empirically determined surface wave spectrum $\Phi(0,\omega)$ may be engendered by linear or nonlinear (large amplitude) surface wave velocity fields. From the point of view of an oceanographer gathering empirical data on surface wave statistics, the precise dynamical description of the velocity fields below the ocean surface may be of secondary interest. However, for the purpose of computing spectra of magnetic fields generated by ocean currents the accuracy of the adopted dynamical model is substantially more important, since the induced magnetic field is proportional to a volume integral over the velocity field. Because no generally agreed upon theory describing nonlinear surface wave phenomena appears available, we are forced to rely on the usual crude linear model, which, strictly speaking, holds only for surface displacements that are infinitely small. Thus, even though we shall express the spectrum of the velocity potential giving rise to surface waves, and the resulting magnetic field spectra, in terms of an "arbitrary" $\Phi_{\eta\eta}(\underline{\rho},\omega)$, the correctness of the results can certainly be no better than the accuracy of the underlying linear dynamical model. In other words, use of more refined models for the temporal (or spatial) spectrum of surface wave displacement in the formulas for magnetic field spectra will not necessarily improve their accuracy.

It is generally assumed that the velocity field giving rise to surface waves is irrotational, so that

$$\underline{V} = \nabla\phi$$

which, together with the incompressibility condition, gives

$$\nabla^2\phi = 0 .$$

One fundamental approximation underlying linear theory is that the ocean surface is nearly flat. For a deep ocean one must have $\phi \to 0$ as $y \to -\infty$, so that the solution of the Laplace equation is

$$\phi(\underline{\rho},y,t) = \iint_{-\infty}^{\infty} e^{k_T y}\, e^{-i\underline{k}_T\cdot\underline{\rho}}\, F(\underline{k}_T,t)\, d^2\underline{k}_T \qquad \text{(A-15)}$$

where $-\infty \leq y \leq 0$.

If the surface displacement is sufficiently small, $F(\underline{k}_T,t)$ in (A-15) may be related to $F(\underline{k}_T,t)$ in (A-1). Thus, for small displacements

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=0} \approx \frac{\partial\eta}{\partial t}$$

and with the aid of (A-1) and (A-15) one obtains

$$\dot{F}(\underline{k}_T,t) = k_T F(\underline{k}_T,t) \qquad \text{(A-16)}$$

where the dot denotes the partial derivative with respect to time. Inserting this in (A-15) yields

$$\phi(\underline{\rho},y,t) = \iint_{-\infty}^{\infty} e^{k_T y}\, e^{-i\underline{k}_T\cdot\underline{\rho}}\, \frac{\dot{F}(\underline{k}_T,t)}{k_T}\, d^2\underline{k}_T \quad . \qquad \text{(A-17)}$$

Finally, with the aid of the linearized momentum equation at the air-water interface one can obtain a differential equation for F. Thus, under the assumption of constant pressure and that capillary* waves may be neglected, one has

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad y = 0 \quad ,$$

or, equivalently,

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \eta}{\partial t} = 0 \quad .$$

After substituting (A-17) and (A-1) in the last relation, we obtain

$$\ddot{F}(\underline{k}_T,t) + k_T g \, F(\underline{k}_T,t) = 0 \quad . \tag{A-18}$$

Aside from a constant (independent of time), which we set equal to zero, the general solution of (A-18) is

$$F(\underline{k}_T,t) = A^+(\underline{k}_T) \, e^{i\Omega t} + A^-(\underline{k}_T) \, e^{-i\Omega t} \quad , \tag{A-19}$$

where Ω is the dispersion relation for deep water surface waves,

$$\Omega = + \sqrt{k_T g} \quad , \tag{A-20}$$

and $A^+(\underline{k}_T)$, $A^-(\underline{k}_T)$ are the two constants of integration. In terms of these two constants the displacement and the velocity potential are

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*The wavelengths of capillary waves are too short to be of interest herein.

$$\eta(\underline{\rho},t) = \iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}} \left[A^+(\underline{k}_T)\, e^{i\Omega t} + A^-(\underline{k}_T)\, e^{-i\Omega t}\right] d^2\underline{k}_T \quad , \tag{A-21}$$

$$\phi(\underline{\rho},y,t) = \iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}}\, e^{k_T y}\, \frac{i\Omega}{k_T}\left[A^+(\underline{k}_T)\, e^{i\Omega t} - A^-(\underline{k}_T)\, e^{-i\Omega t}\right] d^2\underline{k}_T \,. \tag{A-22}$$

Taking the gradient of (A-22) gives the velocity field $\underline{V}(\underline{r},t)$:

$$\underline{V}(\underline{r},t) = \iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}}\ \hat{\underline{V}}(\underline{k}_T,y,t)\ d^2\underline{k}_T \quad , \tag{A-23}$$

where

$$\hat{\underline{V}}(\underline{k}_T,y,t) = e^{k_T y}\, \Omega k_T^{-1} \left[A^+(\underline{k}_T)\, e^{i\Omega t} - A^-(\underline{k}_T)\, e^{-i\Omega t}\right](\underline{k}_T + i\underline{y}_o\, k_T). \tag{A-24}$$

We should now like to relate the statistical averages of the spectral amplitudes A^+, A^- to the spectra $\Phi_{\eta\eta}$ and $S_{\eta\eta}$ in (A-11) and (A-5). The correlation function of $F(\underline{k}_T,t)$ in (A-19) in time-wave number space is

$$\langle F(\underline{k}_T,t_1)\, F^*(\underline{k}_T',t_2)\rangle$$

$$= \langle A^+(\underline{k}_T)\, A^{*+}(\underline{k}_T')\rangle\, e^{i\Omega t_1 - i\Omega' t_2} + \langle A^-(\underline{k}_T)\, A^{*-}(\underline{k}_T')\rangle\, e^{-i\Omega t_1 + i\Omega' t_2}$$

$$+ \langle A^-(\underline{k}_T)\, A^{*+}(\underline{k}_T')\rangle\, e^{-i\Omega t_1 - i\Omega' t_2} + \langle A^+(\underline{k}_T)\, A^{*-}(\underline{k}_T')\rangle\, e^{i\Omega t_1 + i\Omega' t_2} \quad , \tag{A-25}$$

where $\Omega = \Omega(k_T)$ and $\Omega' = \Omega(k_T')$.

In order that this expression reduce to the form $S_{\eta\eta}(\underline{k}_T,t_1-t_2)\ \delta(\underline{k}_T-\underline{k}_T')$, as required by a spatially homogeneous and stationary process, it is necessary that the following relations hold:

$$\langle A^{+}(\underline{k}_T)\, \overset{*}{A}{}^{+}(\underline{k}_T')\rangle = \frac{1}{2}\,\psi^{+}(\underline{k}_T)\,\delta(\underline{k}_T-\underline{k}_T') \quad , \tag{A-26a}$$

$$\langle A^{-}(\underline{k}_T)\, \overset{*}{A}{}^{-}(\underline{k}_T')\rangle = \frac{1}{2}\,\psi^{-}(\underline{k}_T)\,\delta(\underline{k}_T-\underline{k}_T') \quad , \tag{A-26b}$$

$$\langle A^{+}(\underline{k}_T)\, \overset{*}{A}{}^{-}(\underline{k}_T')\rangle = \langle A^{-}(\underline{k}_T)\, \overset{*}{A}{}^{+}(\underline{k}_T')\rangle = 0 \quad , \tag{A-26c}$$

where ψ^{+} and ψ^{-} are two as yet unspecified functions of $\underline{k}_T$. With $t_1 - t_2 = \tau$, (A-26) inserted in (A-25) yields $S_{\eta\eta}(\underline{k}_T,\tau)\,\delta(\underline{k}_T-\underline{k}_T')$, while the explicit form of the spatial cross-spectrum in terms of ψ^{+} and ψ^{-} becomes

$$S_{\eta\eta}(\underline{k}_T,\tau) = \frac{1}{2}\left[\psi^{+}(\underline{k}_T)\, e^{i\Omega\tau} + \psi^{-}(\underline{k}_T)\, e^{-i\Omega\tau}\right] \quad . \tag{A-27}$$

The functions $\psi^{+}(\underline{k}_T)$ and $\psi^{-}(\underline{k}_T)$ must be compatible with (A-8) and (A-9). The first of these requires that,

$$\psi^{+}(\underline{k}_T) = \overset{*}{\psi}{}^{-}(-\underline{k}_T) \ ,$$

$$\psi^{-}(\underline{k}_T) = \overset{*}{\psi}{}^{+}(-\underline{k}_T) \ ,$$

while according to (A-9) ψ^{+} and ψ^{-} must be purely real functions. Consequently, $\psi^{+}(\underline{k}_T) = \psi^{-}(-\underline{k}_T)$ and the spatial cross-spectrum may be written in terms of the single real positive function $\psi(\underline{k}_T) \equiv \psi^{+}(\underline{k}_T)$. We then have

$$S_{\eta\eta}(\underline{k}_T,\tau) = \frac{1}{2}\left[\psi(\underline{k}_T)\, e^{i\Omega\tau} + \psi(-\underline{k}_T)\, e^{-i\Omega\tau}\right] \quad , \tag{A-28}$$

and the correlation function of the displacement becomes

$$R_{\eta\eta}(\underline{\rho},\tau) = \frac{1}{2}\iint_{-\infty}^{\infty} e^{-i\underline{k}_T\cdot\underline{\rho}}\left[\psi(\underline{k}_T)\, e^{i\Omega\tau} + \psi(-\underline{k}_T)\, e^{-i\Omega\tau}\right] d^2\underline{k}_T \ . \tag{A-29}$$

We now obtain the relationship between $\psi(\underline{k}_T)$ and the temporal cross-spectrum. Taking the Fourier transform of (A-29) with respect to τ yields

$$\Phi(\underline{\rho},\omega) = \pi \iint_{-\infty}^{\infty} e^{-i\underline{k}_T \cdot \underline{\rho}} \left[\psi(\underline{k}_T)\ \delta(\omega-\Omega) + \psi(-\underline{k}_T)\ \delta(\omega+\Omega)\right] d^2\underline{k}_T . \quad \text{(A-30)}$$

It will be convenient to introduce polar coordinates $k_x = k_T \cos w$, $k_z = k_T \sin w$, $x = \rho \cos\theta$, $y = \rho \sin\theta$, and employ the notation

$$\psi(\underline{k}_T) = \psi(k_T, w) \quad ,$$

$$\psi(-\underline{k}_T) = \psi(k_T, w+\pi) \quad .$$

Recalling that in the dispersion relationship k_T has been defined only for positive Ω, viz., $\Omega = \sqrt{k_T g} > 0$, we have, for $\omega > 0$,

$$\Phi(\underline{\rho},\omega) = \frac{2\pi\omega^3}{g^2} \int_0^{2\pi} dw\, e^{-i\frac{\omega^2}{g} \rho \cos(w-\theta)}\, \psi\left(\frac{\omega^2}{g}, w\right) \quad . \qquad \text{(A-31)}$$

For $\omega < 0$ only the second delta function in (A-30) contributes, and is to be evaluated at $\omega = -\sqrt{k_T g}$. One then obtains

$$\Phi(\underline{\rho},\omega) = -\frac{2\pi\omega^3}{g^2} \int_0^{2\pi} dw\, e^{-i\frac{\omega^2}{g} \rho \cos(w-\theta)}\, \psi\left(\frac{\omega^2}{g}, w+\pi\right) \quad . \qquad \text{(A-32)}$$

Assuming that ψ is defined for all $0 \leq w \leq 2\pi$ as a single valued function, i.e., $\psi\left(\frac{\omega^2}{g}, 0\right) = \psi\left(\frac{\omega^2}{g}, 2\pi\right)$, the periodicity of $\cos(w-\theta)$ in w permits the replacement of the limits of integration by $\int_{\delta}^{2\pi+\delta}$, where δ is any real quantity. Consequently, changing $w + \pi$ to w in (A-32), one obtains

$$\Phi(\underline{\rho},\omega) = -\frac{2\pi\omega^2}{g^2} \int_0^{2\pi} dw\, e^{i\frac{\omega^2}{g} \rho \cos(w-\theta)}\, \psi\left(\frac{\omega^2}{g}, w\right); \quad \omega < 0 \quad . \qquad \text{(A-33)}$$

Comparing (A-31) with (A-33) one observes that the symmetry conditions stipulated by (A-12) and (A-13) are indeed satisfied. Setting $\underline{\rho} = 0$ in (A-31) and (A-33) we find the spectral density:

$$\Phi(0,\omega) = \frac{2\pi|\omega|^3}{g^2}\int_0^{2\pi} dw\ \psi\left(\frac{\omega^2}{g},w\right) \quad ; \ |\omega|<\infty \ . \qquad \text{(A-34)}$$

In most discussions of ocean surface wave spectra only the positive frequencies are mentioned explicitly. As long as the observation platform is stationary with respect to the wave motion, the negative frequency region may be ignored. However, when the observation platform is moving, the spectrum measured with respect to the platform will undergo a Doppler-like translation and distortion involving positive and negative frequency constituents of $\Phi(0;\omega)$. This is discussed in Chapter VII, and in Appendix E in connection with towed internal wave spectra.

An analytical form of the sea surface displacement spectrum that has received some experimental confirmation is the Pierson-Neumann spectrum. (Kinsman [11], pp. 386.) In terms of the amplitude function A^2 employed by Kinsman, p. 399, Eq. 8.4:15, the functional form of this spectrum is

$$A^2(\omega,w) = \begin{cases} \overline{C}\,\omega^{-6} \exp\{-2g^2\omega^{-2}U^{-2}\}\cos^2(w-w_0); \ \omega_1 \le \omega < \infty, -\frac{\pi}{2} < w-w_0 < \frac{\pi}{2} \ , \\ 0 \ ; \ \text{otherwise} \ . \end{cases} \qquad \text{(A-35)}$$

where U is the wind speed, w_0 the wind direction. We have replaced σ and θ, used by Kinsman, with Ω and $w-w_0$, respectively. The constant $\overline{C}$ has the numerical value [Kinsman, p. 390]

$$\overline{C} = 3.05\ \mathrm{m^2 sec^{-5}} \ . \qquad \text{(A-36)}$$

The quantity ω_I is the low-frequency cutoff which is determined by the fetch and wind duration. When $\omega_I = 0$ the sea is said to be fully aroused. Data for determining ω_I for a given wind speed fetch and duration may be found in Kinsman, p. 396. Next we should like to relate $A^2(\omega,w)$ of Kinsman to $\psi\left(\frac{\omega^2}{g},w\right)$.

In our notation, Eq. 8.3:6 on p. 380 of Kinsman for the correlation of the ocean surface displacement reads

$$R_{\eta\eta}(0,\tau) = \frac{1}{2}\int_0^\infty \left\{\int_0^{2\pi} A^2(\omega,w)\, dw\right\} \cos \omega\tau \, d\omega \quad . \tag{A-37}$$

On the other hand [see (A-11)],

$$R_{\eta\eta}(0,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(0,\omega)\, e^{i\omega\tau}\, d\omega \, .$$

Substituting from (A-34) one obtains

$$R_{\eta\eta}(0,\tau) = \int_{-\infty}^{\infty} \frac{|\omega|^3}{g}\, e^{i\omega\tau} d\omega \int_0^{2\pi} dw\, \psi\left(\frac{\omega^2}{g},w\right)$$

$$= \frac{1}{2}\int_0^\infty \left\{\int_0^{2\pi} \frac{4\omega^3}{g}\, \psi\left(\frac{\omega^2}{g},w\right) dw\right\} \cos \omega\tau \, d\omega \, .$$

Comparing the last expression with (A-37) one has

$$A^2(\omega,w) = \frac{4\omega^3}{g^2}\, \psi\left(\frac{\omega^2}{g},w\right) \quad .$$

Referring to (A-35), the explicit expression for ψ is

$$\psi\left(\frac{\omega^2}{g},w\right) = \begin{cases} \frac{\bar{C}}{4}\, \omega^{-5} g^2 \exp\{-2g^2\omega^{-2}U^{-2}\} \cos^2(w-w_0); & \omega_I < \omega < \infty \\ & -\frac{\pi}{2} < w - w_0 < \frac{\pi}{2} \\ 0 \;;\; \text{otherwise} \;, & \end{cases} \tag{A-38}$$

The most uncertain feature in the Pierson-Neumann spectrum is wave number directionality. The available data appear so crude as to be compatible with a variety of functional forms. (Kinsman [11], p. 401.)

APPENDIX B

EVALUATION OF CERTAIN CONVOLUTION TYPE INTEGRALS INVOLVING THE FREE SPACE GREEN'S FUNCTION

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Combining (48) and (47) we have

$$\phi'_z(x,y,z) = B_{oz}\int_{-\infty}^{0}dy'\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dz'\ K_z(x,y,z;x',y',z')\ \omega_y(x',y',z')\ , \tag{B-1}$$

where

$$K_z(x,y,z;x',y',z')$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx''\int_{-\infty}^{\infty}dz''\ [(x-x'')^2+(z-z'')^2+y^2]^{-\frac{1}{2}}\ \frac{\partial}{\partial z''}G_N(x'',0,z'';x',y',z') \tag{B-2}$$

and G_N is defined in (26).

We shall employ the Fourier integral representation of the free space Green's function G_o :

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$$G_o(x,y,z;x',y',z') \equiv \frac{1}{4\pi}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-\frac{1}{2}} \equiv \frac{1}{4\pi}\left|\underline{r}-\underline{r}'\right|^{-1}$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta \; e^{-i\xi(x-x') -i\zeta(z-z')} \frac{e^{-k_T|y-y'|}}{2k_T} , \qquad \text{(B-3)}$$

where

$$k_T = \sqrt{\xi^2 + \zeta^2} \; . \qquad \text{(B-4)}$$

We then have

$$\frac{1}{2\pi}\left[(x-x'')^2 + (z-z'')^2 + y^2\right]^{-\frac{1}{2}}$$

$$= \frac{2}{(2\pi)^2}\iint_{-\infty}^{\infty} d\xi d\zeta \; e^{-i\xi(x-x'') -i\zeta(z-z'')} \frac{e^{-k_T|y|}}{2k_T} \; . \qquad \text{(B-5)}$$

Also (see (27)),

$$\frac{\partial}{\partial z''} G_N(x'',0,z'';x',y',z') \equiv \frac{1}{2\pi} \frac{\partial}{\partial z''}[(x''-x')^2 + (z''-z')^2 + y'^2]^{-\frac{1}{2}}$$

$$= \frac{2}{(2\pi)^2} \iint_{-\infty}^{\infty} d\xi' d\zeta' \, e^{-i\xi'(x''-x') - i\zeta'(z''-z')} \frac{(-i\zeta') \, e^{-k_T|y'|}}{2k_T'}$$

(B-6)

After (B-5) and (B-6) are substituted in (B-2) the integration over x" and z" yields a product of delta functions with the result

$K_z(x,y,z;x',y',z')$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d\xi d\zeta \iint_{-\infty}^{\infty} d\xi' d\zeta'$$

$$e^{-i\xi x + i\xi' x' - i\zeta z + i\zeta' z'} \frac{(-i\zeta') \, e^{-k_T|y| - k_T'|y'|}}{k_T' k_T} \delta(\xi-\xi') \, \delta(\zeta-\zeta')$$

Integrating with respect to ξ', ζ' yields

$K_z(x,y,z;x',y',z')$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d\xi d\zeta \, e^{-i\xi(x-x') - i\zeta(z-z')} (-i\zeta) \frac{e^{-k_T(|y|+|y'|)}}{k_T^2}$$

Since y' is always non-positive, for $y > 0$

$$|y|+|y'| = y-y' = |y-y'| .$$

On the other hand, for $y < 0$,

$$|y|+|y'| = -y-y' = -(y+y') = |y+y'| .$$

Hence,

$$K_z(x,y,z;x',y',z') =$$

$$\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d\xi d\zeta \; e^{-i\xi(x-x') -i\zeta(z-z')}(-i\zeta) \frac{e^{-k_T|y\mp y'|}}{k_T^2} . \qquad \text{(B-7)}$$

Let

$$\xi = k_T \cos w$$

$$\zeta = k_T \sin w$$

$$x-x' = |\underline{\rho}-\underline{\rho}'| \cos \theta$$

$$z-z' = |\underline{\rho}-\underline{\rho}'| \sin \theta .$$

Then (B-7) becomes

$K_z(x,y,z;x',y',z')$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dk_T\, e^{-k_T|y\mp y'|} \int_0^{2\pi} dw \sin w\, e^{-ik_T|\underline{\rho}-\underline{\rho}'|\cos(w-\theta)} . \quad \text{(B-8)}$$

We now focus on the inner integral

$$I \equiv \int_0^{2\pi} dw \sin w\, e^{-ik_T|\underline{\rho}-\underline{\rho}'|\cos(w-\theta)}$$

$$= -\int_{-\pi-\theta}^{\pi-\theta} dw' \sin(w'+\theta)\, e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos w'}$$

$$= -\int_0^{2\pi} dw \sin(w+\theta)\, e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos w} , \quad \text{(B-9)}$$

where we have first changed the variable of integration to $w' = w - \theta - \pi$ and then used the observation that the integral of a continuous periodic function taken over a full period is independent of the location of the integration interval. Some further rewriting of the last expression in (B-9) gives

$$I = -\int_0^{2\pi} dw \left(\frac{e^{i(w+\theta)} - e^{-i(w+\theta)}}{2i}\right) e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos w}$$

$$= -\frac{e^{i\theta}}{2i}\int_0^{2\pi} dw\ e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos w}\ e^{iw}$$

$$+ \frac{e^{-i\theta}}{2i}\int_0^{2\pi} dw\ e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos w}\ e^{-iw}$$

$$= -\frac{e^{i\theta}}{2i} e^{i\pi/2}\int_0^{2\pi} dw\ e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos w}\ e^{i(w-\pi/2)}$$

$$+ \frac{e^{-i\theta}}{2i}\int_{-2\pi}^{0} dw\ e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos w}\ e^{iw}$$

$$= -\frac{e^{i\theta}}{2i} e^{i\pi/2}\int_0^{2\pi} dw\ e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos w}\ e^{i(w-\pi/2)}$$

$$+ \frac{e^{-i\theta}e^{i\pi/2}}{2i}\int_0^{2\pi} dw\ e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos w}\ e^{i(w-\pi/2)}$$

$$= -i\sin\theta\int_0^{2\pi} dw\ e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos w}\ e^{i(w-\pi/2)}$$

$$= -2\pi i\sin\theta\ J_1(k_T|\underline{\rho}-\underline{\rho}'|)\ ,$$

where in the last step we have employed the definition of the Bessel function of order 1. We now substitute this result into (B-8) to obtain

$$K_z(x,y,z;x',y',z') = \frac{-\sin\,\theta}{2\pi}\int_0^{\infty} dk_T\; e^{-k_T|y\mp y'|}J_1(k_T|\underline{\rho}-\underline{\rho}'|). \quad \text{(B-10)}$$

From a well-known formula*

$$\int_0^{\infty} e^{-\alpha x}\; J_n(\beta x)\; dx = \frac{\beta^{-n}[\sqrt{\alpha^2+\beta^2}-\alpha]^n}{\sqrt{\alpha^2+\beta^2}} \quad .$$

With $n = 1$, $\alpha = |y\mp y'|$, $\beta = |\underline{\rho}-\underline{\rho}'|$,

$$K_z(x,y,z;x',y',z') = -\frac{\sin\,\theta\;(\sqrt{|\underline{\rho}-\underline{\rho}'|^2+(y\mp y')^2}-|y\mp y'|)}{2\pi\;|\underline{\rho}-\underline{\rho}'|\sqrt{|\underline{\rho}-\underline{\rho}'|^2+(y\mp y')^2}} \quad .$$

We now express $\sin\theta$ in terms defined following (B-7), and write the final result

*I. S. Gradsteyn and I. M. Ryzhik, "Table of Integrals, Series and Products", Academic Press, London (1965) p. 707 formula 6.611.

$K_z(x,y,z;x',y',z')$

$$= -\frac{1}{2\pi} \frac{(z-z')\left(\sqrt{|\underline{\rho}-\underline{\rho}'|^2 + (y\mp y')^2} - |y\mp y'|\right)}{|\underline{\rho}-\underline{\rho}'|^2 \sqrt{|\underline{\rho}-\underline{\rho}'|^2 + (y\mp y')^2}} , \qquad \text{(B-11)}$$

where $y\mp y'$ pertains to $y > 0$ and $y < 0$, respectively.

In a similar fashion, we evaluate K_y in (57). Substitution of (48) in (56) yields

$K_y(x,y,z;x',y',z')$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dz'' \frac{\partial}{\partial y} [(x-x'')^2 + (z-z'')^2 + y^2]^{-\frac{1}{2}} G_N(x'',0,z'';x',y',z').$$

(B-12)

Since only the case $y > 0$ is of interest, we obtain, with the aid of (B-3),

$$\frac{1}{2\pi} \frac{\partial}{\partial y} [(x-x'')^2 + (z-z'')^2 + y^2]^{-\frac{1}{2}}$$

$$= \frac{-2}{(2\pi)^2} \iint_{-\infty}^{\infty} d\xi d\zeta \; e^{-i\xi(x-x'') - i\zeta(z-z'')} \frac{e^{-k_T y}}{2} . \qquad \text{(B-13)}$$

Also, by omitting the factor $(-i\zeta')$ in (B-6) one has

$$G_{yy}(x'',0,z'';x'',y',z')$$

$$= \frac{2}{(2\pi)^2} \int_{-\infty}^{\infty} d\xi' d\zeta' \, e^{-i\xi'(x''-x') - i\zeta'(z''-z')} \frac{e^{-k_T' |y'|}}{2k_T'} \; . \quad \text{(B-14)}$$

Substitution of (B-14) and (B-13) in (B-12) again permits the resulting delta function $(2\pi)^2 \, \delta(\zeta-\zeta') \, \delta(\xi-\xi')$ to be integrated out, so that by analogy with the previous case, one obtains

$$K_y(x,y,z;x',y',z') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d\xi d\zeta \, e^{-i\xi(x-x') - i\zeta(z-z')} \frac{e^{-k_T(y-y')}}{k_T} \; ,$$

(B-15)

where we have set $y + |y'| = y - y'$ since $y' < 0$ while $y > 0$. Changing to polar coordinates (via the relations following B-7), (B-15) becomes

$$K_y(x,y,z;x',y',z') = -\int_0^{\infty} dk_T \, e^{-k_T(y-y')} \frac{1}{(2\pi)^2} \int_0^{2\pi} dw \, e^{ik_T|\underline{\rho}-\underline{\rho}'|\cos(w-\theta)} \; .$$

The inner integral is obviously independent of θ and, in fact, defines $2\pi\, J_o(k_T|\underline{\rho}-\underline{\rho}'|)$. Consequently,

$$K_y(x,y,z;x',y',z') = -\frac{1}{2\pi}\int_0^\infty dk_T\; e^{-k_T(y-y')}\; J_o(k_T|\underline{\rho}-\underline{\rho}'|)\;. \quad \text{(B-16)}$$

Using the same formula as in the evaluation of (B-10) we find

$$K_y(x,y,z;x',y',z') = -\frac{1}{2\pi}\,[(x-x')^2\;(y-y')^2\;+(z-z')^2]\;, \quad \text{(B-17)}$$

as was to be demonstrated.

APPENDIX C

FORMULATION FOR ELECTROSTATIC AND MAGNETOSTATIC FIELDS IN TERMS OF THE LORENTZ POTENTIAL

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FORMULATION FOR ELECTROSTATIC AND MAGNETOSTATIC FIELDS IN TERMS OF THE LORENTZ POTENTIAL

Here we present the derivation of the expressions for the matrix elements given in Eq. (70).

$$\nabla \times \underline{E} = 0 \tag{C-1}$$

$$\nabla \times \underline{H} = \begin{cases} 0 \quad ; \quad y > 0 & \text{(C-2a)} \\ \sigma\, \underline{E} + \sigma\, \underline{V} \times \underline{B}_o \quad ; \quad y < 0 & \text{(C-2b)} \end{cases}$$

$$\mu_o\, \underline{H} = \nabla \times \underline{A} \qquad \underline{E} = -\nabla\phi$$

$$\nabla \times \nabla \times \underline{A} = \begin{cases} 0 \quad ; \quad y > 0 \\ -\mu_o\sigma\nabla\phi + \mu_o\sigma(\underline{V} \times \underline{B}_o) \end{cases}$$

$$\nabla \times \nabla \times \underline{A} = \nabla\nabla\cdot\underline{A} - \nabla^2\underline{A} \ .$$

Let
$$\nabla\cdot\underline{A} = \begin{cases} 0 \quad ; \quad y > 0 & \text{(C-3a)} \\ -\mu_o\sigma\phi \quad y < 0\ . & \text{(C-3b)} \end{cases}$$

Then
$$\underline{E} = \begin{cases} \dfrac{1}{\sigma\mu_o}\nabla\nabla\cdot\underline{A} \quad ; \quad y < 0 & \text{(C-4a)} \\ -\nabla\phi \quad ; \quad y > 0\ , & \text{(C-4b)} \end{cases}$$

while

$$\nabla^2 \underline{A} = \begin{cases} -\mu_o \sigma (\underline{V} \times \underline{B}_o) \;;\; y < 0 \\ 0 \;;\; y > 0 \;. \end{cases} \tag{C-5}$$

Boundary conditions on $\underline{A}$: Since $\underline{H}$ is finite at $y = 0$ $\nabla \times \underline{A} = \mu_o \underline{H}$ implies continuity of A_x , A_z. Also $\nabla \cdot \underline{A}$ is finite. Therefore A_y must be continuous at $y = 0$. Moreover from (C-2), (since the right side is finite) H_x, H_z are continuous. However,

$$\mu_o H_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$\mu_o H_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \quad ,$$

and we have just concluded that A_y is continous at $y = 0$. From this follows that all derivatives taken *along* the place $y = 0$ must be continuous at $y = 0$—in particular $\frac{\partial A_y}{\partial z}$ and $\frac{\partial A_x}{\partial x}$ are then continuous. Continuity of H_x and H_z then implies continuity of $\frac{\partial A_z}{\partial y}$ and $\frac{\partial A_x}{\partial y}$. In addition, we must have from (C-2),

$$E_y + \underline{y}_o \cdot (\underline{V} \times \underline{B}_o) = 0 \quad \text{at} \quad y = 0^-$$

(continuity of total current across the interface) or employing (C-4a)

$$\frac{\partial}{\partial y} \nabla \cdot \underline{A} + \sigma\mu_o \, \underline{y}_o \cdot (\underline{V} \times \underline{B}_o) = 0 \text{ at } y = 0^- . \tag{C-6}$$

On the other hand, from (C-5)

$$\nabla^2 A_y = -\mu_o \sigma \, \underline{y}_o \cdot \underline{V} \times \underline{B}_o \; ; \; y < 0 . \tag{C-8}$$

Consequently, the fluid velocity dependent term in (C-6) may be eliminated to obtain

$$\frac{\partial}{\partial y} (\nabla \cdot \underline{A}) - \nabla^2 A_y = 0 \text{ at } y = 0^- ,$$

or, equivalently,

$$\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2} = \frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_z}{\partial z} \right) \text{ at } y = 0^- . \tag{C-9}$$

Equation (C-9) together with

$$A_x\Big|_{y=0^-} = A_x\Big|_{y=0^+} \,, \quad \frac{\partial A_x}{\partial y}\Big|_{y=0^-} = \frac{\partial A_x}{\partial y}\Big|_{y=0^+}$$

(C-10)

$$A_z\Big|_{y=0^-} = A_z\Big|_{y=0^+} \,, \quad \frac{\partial A_z}{\partial y}\Big|_{y=0^-} = \frac{\partial A_z}{\partial y}\Big|_{y=0^+}$$

(C-11)

$$A_y\Big|_{y=0^-} = A_y\Big|_{y=0^+} \qquad \text{(C-12)}$$

represents all the boundary conditions needed to solve (C-5).

For convenience, let

$$\sigma\, \underline{V}(x,y,z) \times \underline{B}_o = \underline{J}^{(s)}(x,y,z) \quad , \qquad \text{(C-13)}$$

and write out the three components of the vector equation (C-5) as follows:

$$\nabla^2 A_z = -\mu_o J_z^{(s)} \,, \qquad \text{(C-14a)}$$

$$\nabla^2 A_x = -\mu_o J_x^{(s)} \,, \qquad \text{(C-14b)}$$

$$\nabla^2 A_y = -\mu_o J_y^{(s)} \,, \qquad \text{(C-14c)}$$

It would appear that continuity of the functions A_z, A_x and their normal derivatives as specified in (C-10) and (C-11) is sufficient to uniquely determine A_z and A_x via (C-14a) and (C-14b), since the problem thus posed is identical to that for a current distribution $\underline{x}_o \, J_x^{(s)} + \underline{z}_o \, J_z^{(s)}$ in free space (i.e., in the absence of boundaries). While undoubtedly this would be the solution to (C-14), it would not necessarily be the correct solution for the vector potential of the magnetostatic problem which requires that (C-9) also be satisfied.

The vector problem requires the simultaneous solution of (C-14) with the boundary conditions (C-9) through (C-12). This problem is best handled by considering one component of $\underline{J}^{(s)}$ at a time. Denote by A_{ij} ($i = x,y,z \quad j = x,y,z$) the i-th component of the vector potential due to the j-th component of source current $\underline{J}^{(s)}$. Thus, for $J_z^{(s)}$ we have

$$\nabla^2 A_{zz} = -\mu_o \, J_z^{(s)} \tag{C-15a}$$

$$\nabla^2 A_{yz} = 0 \tag{C-15b}$$

$$\nabla^2 A_{xz} = 0 \; . \tag{C-15c}$$

From (C-9) through (C-12) we observe that A_{xz} is superfluous, so that we can set

$$A_{xz} = 0 \; , \tag{C-16}$$

and we have to deal with only two components A_{zz} , A_{yz} with the boundary conditions

$$A_{zz}\Big|_{y=0^-} = A_{zz}\Big|_{y=0^+} \tag{C-17a}$$

$$\frac{\partial A_{zz}}{\partial y}\Big|_{y=0^-} = \frac{\partial A_{zz}}{\partial y}\Big|_{y=0^+} \tag{C-17b}$$

$$A_{yz}\Big|_{y=0^-} = A_{yz}\Big|_{y=0^+} \tag{C-17c}$$

$$\frac{\partial^2 A_{yz}}{\partial x^2} + \frac{\partial^2 A_{yz}}{\partial z^2} = \frac{\partial^2 A_{zz}}{\partial y \partial z}, \text{ at } y = 0^- \quad . \tag{C-17d}$$

The solution for A_{zz} is

$$A_{zz}(x,y,z) = \mu_o \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2 \underline{\rho}' \; G_o(\underline{r},\underline{r}') \; J_z^{(s)} (\underline{r}') \; , \tag{C-18}$$

where the free space Green's function obviously satisfies the same boundary conditions as A_{zz}; A_{yz} must also be linearly related to $\mu_o \; J_z^{(s)}$, and we can always write

$$A_{yz}(x,y,z) = \mu_o \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2\underline{\rho}'\, G_{yz}(\underline{r},\underline{r}')\, J_z^{(s)}(\underline{r}') \ , \tag{C-19}$$

where G_{yz} is an unknown function to be determined from the boundary conditions at $y = 0$. Since G_{yz} satisfies the homogeneous Laplace equation (just as does A_{yz}), we can always write it in the form

$$G_{yz}(\underline{r},\underline{r}') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2\underline{k}_T\, e^{-i\underline{k}_T\cdot(\underline{\rho}-\underline{\rho}')}\, g_{yz}(\underline{k}_T,y,y') \tag{C-20}$$

where of course $y' < 0$, and

$$g_{yz} = \begin{cases} \dfrac{T}{2k_T}\, e^{k_T y' - k_T y} & ;\ y > 0 \qquad \text{(C-21a)} \\[2ex] \dfrac{T}{2k_T}\, e^{k_T y' + k_T y} & ;\ y < 0 \ . \qquad \text{(C-21b)} \end{cases}$$

This form ensures that g_{yz} (and hence G_{yz}) is continuous at $y = 0$ and twice differentiable with respect to y for all $y \neq 0$. (G_{yz} satisfies the homogeneous Laplace equation.) The single unknown coefficient $T(\underline{k}_T)$ is found by writing $G_o(\underline{r},\underline{r}')$ in the Fourier domain and employing (C-17d). Thus, with

$$G_o(\underline{r},\underline{r}') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2\underline{k}_T\ e^{-i\underline{k}_T\cdot(\underline{\rho}-\underline{\rho}')}\ \frac{e^{-k_T|y-y'|}}{2k_T} \ ,$$

one finds

$$\frac{T}{2k_T} e^{k_T y'} \left(-k_T^2 \right) = \frac{ik_z}{2} e^{k_T k'}$$

or

$$T = \frac{-ik_z}{k_T} \quad . \tag{C-22}$$

Hence,

$$G_{yz}(\underline{r},\underline{r}') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2\underline{k}_T \; e^{-i\underline{k}_T \; (\underline{\rho}-\underline{\rho}')} \frac{(-ik_z) \; e^{-k_T|y\mp y'|}}{2k_T^2} \tag{C-23}$$

where the minus and plus sign refers to y > 0 and y < 0, respectively.

This integral has been evaluated in Appendix B, Eq. (B-11):

$$G_{yz}(\underline{r},\underline{r}') = -\frac{1}{4\pi} \frac{(\underline{z}-\underline{z}') \left(\sqrt{|\underline{\rho}-\underline{\rho}'|^2 + (y\mp y)^2} - |y\mp y'|\right)}{|\underline{\rho}-\underline{\rho}'|^2 \sqrt{|\underline{\rho}-\underline{\rho}'|^2 + (y\mp y')^2}} \quad . \tag{C-24}$$

Next, we let $\underline{J}^{(s)} = \underline{X}_o J_x^{(s)}$, and solve

$$\nabla^2 A_{xx} = -\mu_o J_x^{(s)} , \qquad \text{(C-26a)}$$

$$\nabla^2 A_{yx} = 0 , \qquad \text{(C-26b)}$$

with

$$A_{zx} = 0 . \qquad \text{(C-26c)}$$

Proceeding as before,

$$A_{xx}(x,y,z) = \mu_o \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2\underline{\rho}' \quad G_o(\underline{r},\underline{r}') \, J_x^{(s)}(\underline{r}') , \qquad \text{(C-27)}$$

$$A_{yx}(x,y,z) = \mu_o \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2\underline{\rho}' \quad G_{yx}(\underline{r},\underline{r}') \, J_x^{(s)}(\underline{r}') , \qquad \text{(C-28)}$$

where

$$G_{yx}(\underline{r},\underline{r}') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2\underline{k}_T \quad e^{-i\underline{k}_T\cdot(\underline{\rho}-\underline{\rho}')} (-ik_x) \frac{e^{-k_T|y\mp y'|}}{2k_T^2} ,$$

(C-29)

which integrates out to

$$G_{yx}(\underline{r},\underline{r}') = -\frac{1}{4\pi}\,\frac{(x-x')\left(\sqrt{(\underline{\rho}-\underline{\rho}')^2 + (y\mp y')^2} - |y\mp y'|\right)}{|\underline{\rho}-\underline{\rho}'|^2\sqrt{|\underline{\rho}-\underline{\rho}'|^2 + (y\mp y')^2}} \quad . \tag{C-30}$$

Finally, we turn to the third component $J_y^{(s)}$. We have

$$\nabla^2 A_{yy} = -\mu_o\, J_y^{(s)} \quad , \tag{C-31}$$

Obviously the B.C. in (C-10) – (C-12) are met with

$$A_{xy} = A_{zy} = 0 \quad , \tag{C-32}$$

and

$$\frac{\partial^2 A_{yy}}{\partial x^2} + \frac{\partial^2 A_{yy}}{\partial z^2} = 0 \quad \text{at } y = 0^- \quad .$$

Moreover, since A_{yy} is required to be continuous at $y = 0$, we conclude that A_{yy} = constant at $y = 0$, which we can always set equal zero. Since $\nabla^2 A_{yy} = 0$ for $y > 0$, we have the result that

$$A_{yy} = 0 \ , \ y \geq 0 \ . \tag{C-33}$$

On the other hand, for $y \leq 0$,

$$A_{yy}(x,y,z) = \mu_o \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2\underline{\rho}' \quad G_{yy}(\underline{r},\underline{r}') \, J_y^{(s)}(\underline{r}') \; , \qquad \text{(C-34)}$$

where

$$G_{yy}(\underline{r},\underline{r}') = G_o(x,y,z;x',y',z') - G_o(x,y,z;x',-y',z'), \qquad \text{(C-35)}$$

which is the Green's function for the Dirichlet problem. In Fourier transform space one has

$$G_{yy}(\underline{r},\underline{r}') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k_T \; e^{-i\underline{k}_T(\underline{\rho}-\underline{\rho}')} \left[\frac{e^{-k_T|y-y'|} - e^{-k_T|y+y'|}}{2k_T} \right].$$

(C-36)

Collecting the preceding results we have

$$A_z(x,y,z) = \mu_o \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2\underline{\rho}' \quad G_o(\underline{r},\underline{r}') \, J_z^{(s)}(\underline{r}') \qquad \text{(C-37a)}$$

$$A_x(x,y,z) = \mu_o \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2\underline{\rho}' \quad G_o(\underline{r},\underline{r}') \, J_x^{(s)}(\underline{r}') \qquad \text{(C-37b)}$$

$$A_y(x,y,z) = \begin{cases} \mu_o \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2\underline{\rho}' \left[G_{yx}(\underline{r},\underline{r}') J_x^{(s)}(\underline{r}') + G_{yz}(\underline{r},\underline{r}') J_z^{(s)}(\underline{r}') \right] \\ \qquad ; y > 0 \\ \mu_o \int_{-\infty}^{0} dy' \iint_{-\infty}^{\infty} d^2\underline{\rho}' \left[G_{yx}(\underline{r},\underline{r}') J_x^{(s)}(\underline{r}') + G_{yz}(\underline{r},\underline{r}') J_z^{(s)}(\underline{r}') \right. \\ \qquad \left. + G_{yy}(\underline{r},\underline{r}') J_y^{(s)}(\underline{r}') \right] \quad ; y < 0 . \end{cases} \tag{C-37c}$$

From (C-37) one notes that whereas the components of the vector potential tangential to the interface are given in terms of the free space Green's function G_o this is not the case for the vertical component. Since any deviation from the "Biot-Savart" type integral applied to $J^{(s)}$ must be due to additional conduction current generated by the induced electric field (i.e., *within* the conducting fluid), such electric fields evidently influence only the value of the y-component of the vector potential).*

The components of the vector potential $\underline{A}$ uniquely determine both the magnetostatic and electrostatic fields below and above the surface. Below the surface the electrostatic potential is given by

$$\phi = -\frac{1}{\mu_o \sigma} \nabla \cdot \underline{A} \tag{C-38}$$

*Note that if one employs the total current to find $\underline{A}$, i.e., $\underline{J} = \sigma\underline{E} + \sigma(\underline{V} \times \underline{B}_o)$, then $\underline{A} = \underline{A}_1 + \underline{A}_2$, $\underline{A}_2 = \mu_o \iiint \underline{J}^{(s)}(\underline{r}') G_o(\underline{r},\underline{r}')\, d^3\underline{r}'$, $\underline{A}_1 = \underline{y}_o\, [A_y - \mu_o \iiint J_y^{(s)}(\underline{r}')\, G_o(\underline{r},\underline{r}')\, d^3\underline{r}']$, with A_y given by Eq. (C-37c), $\underline{A}_1$ being the vector potential contributed by the electric field induced current $\sigma\underline{E}$.

Continuity of ϕ at $y = 0$ may now be employed to obtain

$$\phi(x,y,z) = -\frac{2}{\mu_o \sigma} \iint_{-\infty}^{\infty} d^2 \underline{\rho}' \frac{\partial}{\partial y} G_o(x,y,z;x',0,z') \nabla' \cdot \underline{A} \ , \tag{C-39}$$

where ∇' denotes differentiation with respect to x',z',y' (at $y' = 0$). One can demonstrate that (C-38) and (C-39) lead to the same results as in Eqs. (29) and (30).

APPENDIX D

FORMULATION FOR INDUCED ELECTRIC AND MAGNETIC FIELDS TAKING ACCOUNT OF DISPLACEMENT CURRENT AND MAGNETIC INDUCTION ABOVE THE OCEAN SURFACE

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FORMULATION FOR INDUCED ELECTRIC AND MAGNETIC FIELDS TAKING ACCOUNT OF DISPLACEMENT CURRENT AND MAGNETIC INDUCTION ABOVE THE OCEAN SURFACE

In Chapter III it was shown on the basis of simple arguments that for time scales on the order of seconds or longer and length scales much less than $10^3 f^{-\frac{1}{2}}$ meters the electromagnetic fields induced by ocean currents are governed by the equations of electrostatics and magnetostatics. In the following, we shall set up the problem exactly, i.e., we shall include magnetic induction effects and the displacement current above the ocean surface. There are at least two reasons for presenting the more detailed analysis. The first is to show rigorously that the exact solutions for the fields reduce to the quasi-static results under the assumptions stated in Chapter III. The second is to obtain a consistent physical picture of electromagnetic energy transfer above the ocean surface. Below the ocean surface we still neglect the displacement current and the convective transport of charge $\rho\underline{V}$, both of which, being proportional to $\varepsilon_o = \frac{1}{36\pi} \times 10^{-9}$, are entirely negligible by comparison with $\sigma\underline{E}$ at frequencies $\sim$1Hz or less. Thus, as the starting point we take Eqs. (73a) and (73b) with $\rho\underline{V} + \varepsilon_o\varepsilon_r \frac{\partial E}{\partial t} = 0$ for $y < 0$. Unlike in the analysis presented in Chapters II and III, we shall deal directly with the electromagnetic fields, without introducing any potential functions. At the outset we take the Fourier transforms of $\underline{E}$,$\underline{H}$, and $\underline{V}$ with respect to time, viz.,

$$\underline{E}(\underline{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \, \tilde{\underline{E}} (\underline{r},\omega) \, d\omega \, , \qquad \text{(D-1a)}$$

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$$\underline{H}(\underline{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\ \underline{\tilde{H}}(\underline{r},\omega)\ d\omega, \tag{D-1b}$$

$$\underline{V}(\underline{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\ \underline{\tilde{V}}(\underline{r},\omega)\ d\omega\ . \tag{D-1c}$$

The electromagnetic field equations for $\underline{\tilde{E}},\underline{\tilde{H}}$, assume the form

$$\nabla \times \underline{\tilde{E}}\ (\underline{r},\omega) = -i\omega\mu_o\underline{\tilde{H}}(\underline{r},\omega)\quad , \tag{D-2a}$$

$$\nabla \times \underline{\tilde{H}}\ (\underline{r},\omega) = \begin{cases} i\omega\varepsilon_o\ \underline{\tilde{E}}(\underline{r},\omega);\ y>0, \\ \sigma\ [\underline{\tilde{E}}(\underline{r},\omega) + \underline{\tilde{V}}(\underline{r},\omega)\times\underline{B}_o];\ y<0. \end{cases} \tag{D-2b}$$

Since the boundary separating the two media is the xz plane, it is natural to attempt to solve (D-2) in terms of bidimensional Fourier transforms with respect to the spatial variables x,z. Using the notation

$$x\underline{x}_o + z\underline{z}_o = \underline{\rho}$$

($\underline{x}_o$, $\underline{z}_o$ Cartesian unit vectors) and $d^2\underline{\rho} \equiv dxdz$

and, similarly for transform variables

$$\underline{x}_o k_x + \underline{z}_o k_z = \underline{k}_T,\ d^2\underline{k}_T \equiv dk_x dk_z\ ,$$

one can represent $\underline{\tilde{E}}(\underline{r},\omega)$, $\underline{\tilde{H}}(\underline{r},\omega)$ by

$$\underline{\tilde{E}}(\underline{r},\omega) = \iint e^{-i\underline{k}_T\cdot\underline{\rho}}\ \underline{E}(\underline{k}_T,y;\omega)d^2\underline{k}_T\quad , \tag{D-3a}$$

$$\underline{\tilde{H}}(\underline{r},\omega) = \iint e^{-i\underline{k}_T\cdot\underline{\rho}}\underline{H}(\underline{k}_T,y;\omega)d^2\underline{k}_T \quad , \tag{D-3b}$$

$$\underline{\tilde{V}}(\underline{r},\omega) = \iint e^{-i\underline{k}_T\cdot\underline{\rho}}\underline{T}(\underline{k}_T,y;\omega)d^2\underline{k}_T \quad . \tag{D-3c}$$

It will be convenient to define

$$\underline{E}(\underline{k}_T,y;\omega) = \begin{cases} \underline{E}^+(\underline{k}_T,y;\omega);y>0, & \text{(D-4a)} \\ \underline{E}^-(\underline{k}_T,y;\omega);y<0, & \text{(D-4b)} \end{cases}$$

and a similar definition for $\underline{H}$. With this notation, substitution of (D-3) into (D-2) yields

$$-i\underline{k}_T \times \underline{E}^+ + \nabla \times \underline{E}^+ = -i\omega\mu_0\underline{H}^+, \tag{D-5a}$$

$$-i\underline{k}_T \times \underline{H}^+ + \nabla \times \underline{H}^+ = i\omega\varepsilon_0\underline{E}^+;\ y>0\ , \tag{D-5b}$$

and

$$-i\underline{k}_T \times \underline{E}^- + \nabla \times \underline{E}^- = -i\omega\mu_0\underline{H}^-, \tag{D-6a}$$

$$-i\underline{k}_T \times \underline{H}^- + \nabla \times \underline{H}^- = \sigma\left[\underline{E}^- + \underline{T} \times \underline{B}_0\right]\ ;\ y<0\ . \tag{D-6b}$$

Eqs. (D-5) and (D-6) can be solved for $\underline{E}^\pm$ and $\underline{H}^\pm$ after recasting them in a form in which the transverse and longitudinal field components are separated. Thus, define

$$\underline{E}^\pm = \underline{E}_T^\pm + \underline{y}_0\, E_y^\pm\ , \tag{D-7a}$$

$$\underline{H}^{\pm} = \underline{H}^{\pm}_{T} + \underline{y}_{o} H^{\pm}_{y} , \tag{D-7b}$$

where $\underline{y}_o$ is the unit vector in the y direction while $\underline{E}^{\pm}_{T}$, $\underline{H}^{\pm}_{T}$ are fields transverse to $\underline{y}_o$, i.e.,

$$\underline{E}^{\pm}_{T} = \underline{x}_o E^{\pm}_{x} + \underline{z}_o E^{\pm}_{z} , \tag{D-8a}$$

$$\underline{H}^{\pm}_{T} = \underline{x}_o H^{\pm}_{x} + \underline{z}_o H^{\pm}_{z} . \tag{D-8b}$$

Since $\underline{E}^{\pm}$, $\underline{H}^{\pm}$ depend only on y, $\underline{k}_T$ and ω, but not on x,z, one can write

$$\nabla \times \underline{E}^{\pm} = \frac{\partial}{\partial y}\left(\underline{y}_o \times \underline{E}^{\pm}_{T}\right) , \tag{D-9a}$$

$$\nabla \times \underline{H}^{\pm} = \frac{\partial}{\partial y}\left(\underline{y}_o \times \underline{H}^{\pm}_{T}\right) , \tag{D-9b}$$

Also,

$$\underline{k}_T \times \underline{E}^{\pm} \equiv \underline{k}_T \times \underline{E}^{\pm}_{T} + \underline{k}_T \times \underline{y}_o E^{\pm}_{y} , \tag{D-10a}$$

$$\underline{k}_T \times \underline{H}^{\pm} \equiv \underline{k}_T \times \underline{H}^{\pm}_{T} + \underline{k}_T \times \underline{y}_o H^{\pm}_{y} \tag{D-10b}$$

Taking account of (D-9) and (D-10) and forming the scalar vector product of each member of (D-5) and (D-6) with $\underline{y}_o$, yields

$$H_y^{+} = \frac{1}{-\omega\mu_o} \underline{k}_T \cdot \left(\underline{y}_o \times \underline{E}_T^{+}\right) , \tag{D-11a}$$

$$E_y^{+} = \frac{1}{-\omega\varepsilon_o} \underline{k}_T \cdot \left(\underline{H}_T^{+} \times \underline{y}_o\right) , \tag{D-11b}$$

$$H_y^{-} = \frac{1}{-\omega\mu_o} \underline{k}_T \cdot \left(\underline{y}_o \times \underline{E}_T^{-}\right) , \tag{D-12a}$$

$$E_y^{-} = -\frac{1}{\sigma} \underline{k}_T \cdot \left(\underline{H}_T^{-} \times \underline{y}_o\right) - \underline{y}_o \cdot \left(\underline{I} \times \underline{B}_o\right) . \tag{D-12b}$$

Eqs. (11) and (12) express the longitudinal field components $E_y^{\pm}$, $H_y^{\pm}$ explicitly in terms of the transverse components $\underline{E}_T^{\pm}$, $\underline{H}_T^{\pm}$. The transverse components in turn can be found from the solution of two ordinary differential equations in y. To see this, form a vector cross product of each member of (D-5) and (D-6) with $\underline{y}_o$. Thus, since by virtue of (D-10),

$$\underline{y}_o \times \left(\underline{k}_T \times \underline{E}^{\pm}\right) \equiv \underline{y}_o \times \left(\underline{k}_T \times \underline{y}_o E_y^{\pm}\right) = \underline{k}_T E_y^{\pm} , \tag{D-13a}$$

$$\underline{y}_o \times \left(\underline{k}_T \times \underline{H}^{\pm}\right) \equiv \underline{y}_o \times \left(\underline{k}_T \times \underline{y}_o H_z^{\pm}\right) = \underline{k}_T H_y^{\pm} , \tag{D-13b}$$

and by virtue of (D-9)

$$\underline{y}_o \cdot \nabla \times \underline{E}^{\pm} = -\frac{\partial \underline{E}_T^{\pm}}{\partial y} , \tag{D-14a}$$

$$\underline{y}_o \cdot \nabla \times \underline{H}^{\pm} = -\frac{\partial \underline{H}_T^{\pm}}{\partial y} , \tag{D-14b}$$

Eqs. (D-5) and (D-6) may be written in the following form:

$$-i\underline{k}_T \; E_y^+ - \frac{\partial \underline{E}_T^+}{\partial y} = i\omega\mu_o \left(\underline{H}_T^+ \times \underline{y}_o\right) , \tag{D-15a}$$

$$-i\underline{k}_T \; H_y^+ - \frac{\partial \underline{H}_T^+}{\partial y} = i\omega\varepsilon_o \left(\underline{y}_o \times \underline{E}_T^+\right) , \tag{D-15b}$$

$$-i\underline{k}_T \; E_y^- - \frac{\partial \underline{E}_T^-}{\partial y} = i\omega\mu_o \left(\underline{H}_T^- \times \underline{y}_o\right) , \tag{D-16a}$$

$$-i\underline{k}_T \; H_y^- - \frac{\partial \underline{H}_T^-}{\partial y} = \sigma\left(\underline{y}_o \times \underline{E}_T^-\right) + \sigma \; \underline{y}_o \times \left(\underline{T} \times \underline{B}_o\right) . \tag{D-16b}$$

The differential equations for $\underline{E}_T^{\pm}$ and $\underline{H}_T^{\pm}$ are obtained by substituting for $E_y^{\pm}$, $H_y^{\pm}$ from (D-11) and (D-12). This yields

$$-\frac{\partial \underline{E}_T^+}{\partial y} = i\omega\mu_o \left[\underline{\underline{1}} - \frac{\underline{k}_T \underline{k}_T}{k_o^2}\right] \cdot \left(\underline{H}_T^+ \times \underline{y}_o\right) , \tag{D-17a}$$

$$-\frac{\partial \underline{H}_T^+}{\partial y} = i\omega\varepsilon_o \left[\underline{\underline{1}} - \frac{\underline{k}_T \underline{k}_T}{k_o^2}\right] \cdot \left(\underline{y}_o \times \underline{E}_T^+\right) , \tag{D-17b}$$

$$-\frac{\partial \underline{E}_T^-}{\partial y} = i\omega\mu_o \left[\underline{\underline{1}} + \frac{\underline{k}_T \underline{k}_T}{i\omega\mu_o\sigma}\right] \cdot \left(\underline{H}^- \times \underline{y}_o\right) - i\underline{k}_T \underline{y}_o \cdot \underline{T} \times \underline{B}_o , \tag{D-18a}$$

$$-\frac{\partial \underline{H}_T^-}{\partial y} = \left[\sigma \; \underline{\underline{1}} + \frac{\underline{k}_T \underline{k}_T}{i\omega\mu_o}\right] \cdot \left(\underline{y}_o \times \underline{E}_T^-\right) + \sigma \cdot \left(\underline{y}_o \times \underline{T} \times \underline{E}_o\right) \tag{D-18b}$$

where $\underline{\underline{1}} = \underline{x}_o\underline{x}_o + \underline{z}_o\underline{z}_o$ is the unit transverse dyadic.

These vector equations can be reduced to scalar equations by employing the basis vectors $\underline{e}$ and $\underline{e} \times \underline{y}_o$ to write the solutions of (D-17) and (D-18) as

$$\underline{E}_T^{\pm} = V_1^{\pm}(\underline{k}_T, y)\, \underline{e} + V_2^{\pm}(\underline{k}_T, y)\, \underline{e} \times \underline{y}_o \tag{D-19a}$$

$$\underline{H}_T^{\pm} = I_1^{\pm}(\underline{k}_T, y)\, (\underline{y}_o \times \underline{e}) + I_2^{\pm}(\underline{k}_T, y)\underline{e}\ , \tag{D-19b}$$

with $$\underline{e} = \frac{\underline{k}_T}{k_T} \qquad (\underline{e} \cdot \underline{e} = 1)\ .$$

Substituting (D-19) into (D-17) yields

$$\underline{e}\left(-\frac{\partial V_1^+}{\partial y}\right) + (\underline{e} \times \underline{y}_o)\left(-\frac{\partial V_2^+}{\partial y}\right) =$$

$$i\omega\mu_o \left[\underline{\underline{1}} - \frac{\underline{k}_T\underline{k}_T}{k_o^2}\right] \cdot \left\{I_1^+ \underline{e} + I_2^+ (\underline{e} \times \underline{y}_o)\right\}$$

$$= i\omega\mu_o \left(1 - \frac{k_T^2}{k_o^2}\right)\underline{e}\, I_1^+ + i\omega\mu_o\, I_2^+ (\underline{e} \times \underline{y}_o) \tag{20a}$$

and

$$(\underline{y}_o \times \underline{e})\left(-\frac{\partial I_1^+}{\partial y}\right) + \underline{e}\left(-\frac{\partial I_2^+}{\partial y}\right)$$

$$= i\omega\varepsilon_o \left[\underline{\underline{1}} - \frac{\underline{k}_T\underline{k}_T}{k_o^2}\right] \cdot \left\{(\underline{y}_o \times \underline{e})\, V_1^+ + \underline{e}\, V_2^+\right\}$$

$$= i\omega\varepsilon_o\, (\underline{y}_o \times \underline{e})\, V_1^+ + i\omega\varepsilon_o\left(1 - \frac{k_T^2}{k_o^2}\right)\underline{e}\, V_2^+\ . \tag{20b}$$

Since the vectors $\underline{e}$ and $\underline{e} \times \underline{y}_o$ are linearly independent, the scalar multiplicative coefficients appearing on both sides of (D-20) may be equated to obtain

$$-\frac{\partial V_1^+}{\partial y} = i\kappa Z_1 I_1^+ , \qquad \text{(D-21a)}$$

$$-\frac{\partial I_1^+}{\partial y} = i\kappa Y_1 V_1^+ , \qquad \text{(D-21b)}$$

$$-\frac{\partial V_2^+}{\partial y} = i\kappa Z_2 I_2^+ , \qquad \text{(D-22a)}$$

$$-\frac{\partial I_2^+}{\partial y} = i\kappa Y_2 V_2^+ , \qquad \text{(D-22b)}$$

where

$$\kappa = \sqrt{k_o^2 - k_T^2} , \qquad \text{(D-23a)}$$

$$Z_1 = \frac{1}{Y_1} = \frac{\kappa}{\omega\varepsilon_o} , \qquad \text{(D-23b)}$$

$$Z_2 = \frac{1}{Y_2} = \frac{\omega\mu_o}{\kappa} . \qquad \text{(D-23c)}$$

Eqs. (D-21) and (D-22) determine the expansion coefficients in (D-19) for $y > 0$. To obtain similar equations for $y < 0$ one must first express the inhomogeneous terms (driving functions) $-ik_T\underline{y}_o \cdot (\underline{T}\times\underline{B}_o)$ and $\sigma\, \underline{y}_o \times (\underline{T} \times \underline{B}_o)$ in terms of the basis vectors $\underline{e}$ and $\underline{e} \times \underline{y}_o$. Thus, one has

$$-ik_T\underline{y}_o \cdot \underline{T} \times \underline{B}_o = -ik_T\underline{e}\,(\underline{y}_o \cdot \underline{T} \times \underline{B}_o) , \qquad \text{(D-24)}$$

and

$$\sigma \underline{y}_o \times (\underline{T} \times \underline{B}_c) = \sigma \left[\underline{y}_o \times (\underline{T} \times \underline{B}_o) \cdot \underline{e} \right] \underline{e}$$

$$+ \sigma \left[\left\{ \underline{y}_o \times (\underline{T} \times \underline{B}_o) \right\} \cdot \left\{ \underline{e} \times \underline{y}_o \right\} \right] \underline{e} \times \underline{y}_o$$

$$= \xi \, \underline{e} + \eta (\underline{e} \times \underline{y}_o) \quad , \tag{D-25}$$

where

$$\xi = \sigma \left[(\underline{T} \cdot \underline{e}) \, B_{oy} - (\underline{B}_o \cdot \underline{e}) \, T_y \right] \quad , \tag{D-26a}$$

$$\eta = \sigma \left\{ \left[\underline{T} \cdot (\underline{e} \times \underline{y}_o) \right] B_{oy} - \left[\underline{B}_o \cdot (\underline{e} \times \underline{y}_o) \right] T_y \right\} . \tag{D-26b}$$

Substituting (D-24),and (D-26) in (D-18) and employing (D-19), yields

$$\underline{e} \left(- \frac{\partial V_1^-}{\partial y} \right) + (\underline{e} \times \underline{y}_o) \left(- \frac{\partial V_2^-}{\partial y} \right) =$$

$$i\omega\mu_o \left[\underline{\underline{1}} + \frac{\underline{k}_T \underline{k}_T}{i\omega\mu_o \sigma} \right] \cdot \left\{ I_1^- \, \underline{e} + I_2^- \, (\underline{e} \times \underline{y}_o) \right\} - i\underline{k}_T \left\{ \underline{y}_o \cdot (\underline{T} \times \underline{B}_o) \right\} \underline{e} \quad , \tag{D-27a}$$

$$(\underline{y}_o \times \underline{e}) \left(- \frac{\partial I_1^-}{\partial y} \right) + \underline{e} \left(- \frac{\partial I_2^-}{\partial y} \right) =$$

$$= \sigma \left[\underline{\underline{1}} + \frac{\underline{k}_T \underline{k}_T}{i\omega\mu_o} \right] \cdot \left\{ V_1^- (\underline{y}_o \times \underline{e}) + V_2^- \, \underline{e} \right\} + \xi \underline{e} - \eta (\underline{y}_o \times \underline{e}) \quad . \tag{D-27b}$$

Again, by equating the coefficients of $\underline{e}$ and $\underline{e} \times \underline{y}_o$, one finds

$$-\frac{dV_1^-}{dy} = \left(i\omega\mu_o + \frac{k_T^2}{\sigma}\right) I_1^- - ik_T\, \underline{y}_o \cdot (\underline{T}x\underline{B}_o) \,, \tag{D-28a}$$

$$-\frac{dI_1^-}{dy} = \sigma V_1^- - \eta(y) \quad , \tag{D-28b}$$

$$-\frac{dV_2^-}{dy} = i\omega\mu_o I_2^- \quad , \tag{D-29a}$$

$$-\frac{dI_2^-}{dy} = \left(\sigma + \frac{k_T^2}{i\omega\mu_o}\right) V_2^- + \xi(y) \quad . \tag{D-29b}$$

To put these equations into the same form as (D-21) and (D-22), let

$$i\omega\mu_o + \frac{k_T^2}{\sigma} = i\kappa^- Z_1^-$$

$$\sigma = i\kappa^-/Z_1^- \quad \text{or} \quad Z_1^- = \frac{i\kappa^-}{\sigma} \quad .$$

The propagation constant κ^- is then found from

$$i\omega\mu_o + \frac{k_T^2}{\sigma} = -\frac{(\kappa^-)^2}{\sigma}$$

or

$$\kappa^- = -i\sqrt{i\omega\mu_o\sigma + k_T^2} \tag{D-30}$$

and

$$Z_1^- = \frac{\sqrt{i\omega\mu_o\sigma + k_T^2}}{\sigma} = \frac{1}{Y_1^-} \quad . \tag{D-31}$$

Similarly,

$$Z_2^- = \frac{\omega\mu_o}{\kappa^-} = \frac{1}{Y_2^-} \quad . \tag{D-32}$$

The desired form of (D-28) and (D-29) is then

$$-\frac{dV_1^-}{dy} = i\kappa^- Z_1^- I_1^- + \zeta(y) \quad , \tag{D-33a}$$

$$-\frac{dI_1^-}{dy} = i\kappa^- Y_1^- V_1^- - \eta(y) \quad , \tag{D-33b}$$

$$-\frac{dV_2^-}{dy} = i\kappa^- Z_2^- I_2^- \quad , \tag{D-34a}$$

$$-\frac{dI_2^-}{dy} = i\kappa^- Y_2^- V_2^- + \xi(y) \quad , \tag{D-34b}$$

where

$$\zeta(y) = -ik_T \underline{y}_o \cdot (\underline{T} \times \underline{B}_o) \quad . \tag{D-35}$$

Equations (D-21), (D-22), (D-33), and (D-34) represent a set of "transmission line" equations with sources $\zeta(y)$, $\eta(y)$, $\xi(y)$ located in the region $y < 0$. Their solution is best obtained by first solving the corresponding Green's function problems. There are three "canonical" problems that must be considered. They are:

<u>Canonical Problem 1.</u> (E-mode current excited by a unit voltage source).

$$\left.\begin{aligned} -\frac{dV_1}{dy} &= i\kappa^- Z_1^- G_1 + \delta(y-y'), \\ -\frac{dG_1}{dy} &= i\kappa^- Y_1^- V_1 \ , \end{aligned}\right\} \quad y \le 0 \quad , \tag{D-36}$$

$$\left.\begin{aligned} -\frac{dV_1}{dy} &= i\kappa Z_1 G_1 \ , \\ -\frac{dG_1}{dy} &= i\kappa Y_1 V_1 \ . \end{aligned}\right\} \quad y \ge 0 \quad . \tag{D-37}$$

Here $\delta(y-y')$ is the Dirac delta function in which $y'<0$, and $G_1 \equiv G_1(y,y')$ is the desired Green's function. It must satisfy the outgoing wave condition for $y>0$, be continuous (together with its derivative) at $y=0$, and satisfy appropriate boundary conditions at the ocean bottom. Data on the constitutive electromagnetic parameters of the ocean floor do not appear to be readily available. Hopefully, the final results for the induced magnetic fields will not turn out to be overly sensitive to the electrical properties of the material below the ocean floor, especially for deep oceans. In order to retain some generality in the subsequent analytical results, the boundary conditions at $y=-D$ (ocean bottom) will be stated in terms of the E and H mode reflection coeffiecients $\overleftarrow{\Upsilon}_1$ and $\overleftarrow{\Upsilon}_2$, respectively. These are readily expressed in terms of the constitutive parameters of the electromagnetic medium filling the space $y<-D$. If, for example, this medium is assumed to extend to $y=-\infty$ with a conductivity σ_D, relative electrical permittivity ε_{rD} and permeability μ_o, then

$$\overleftarrow{\Upsilon}_1 = \frac{\dfrac{\kappa_D}{\omega\varepsilon_o\varepsilon_{rD}-i\sigma_D} - Z_1^-}{\dfrac{\kappa_D}{\omega\varepsilon_o\varepsilon_{rD}-i\sigma_D} + Z_1^-} , \tag{D-38}$$

and

$$\overleftarrow{\Upsilon}_2 = \frac{\dfrac{\omega\mu_o}{\kappa_D} - Z_2^-}{\dfrac{\omega\mu_o}{\kappa_D} + Z_2^-} . \tag{D-39}$$

where

$$\kappa_D = \sqrt{k_o^2 \left[\epsilon_{rD} - \frac{i\sigma_D}{\omega\epsilon_o}\right]^2 - k_T^2} \quad . \tag{D-40}$$

Canonical Problem 1 can also be phrased in terms of a single second order differential equation. Thus, differentiation of the second equation in (D-36) and (D-37), and substitution of $-\frac{dV_1}{dy}$ yields

$$\frac{d^2G_1}{dy^2} + (\kappa^-)^2 G_1 = i\kappa^- Y_1^- \delta(y-y') \quad , \quad y<0 \tag{D-41}$$

$$\frac{d^2G_1}{dy^2} + \kappa^2 G_1 = 0 \quad , \quad y>0 \quad . \tag{D-42}$$

The solution of (D-41) for y>0 is

$$G_1(y,y') = T(y')e^{-i\kappa y} \quad , \tag{D-43}$$

where $T(y')$ is to be determined from the solution of (D-41), which may be written in the following form

$$G_1(y,y') = A\, \overleftarrow{f(y_<)}\, \overrightarrow{f(y_>)} \quad . \tag{D-44}$$

The symbol $y_<$ denotes the smaller of y or y' while $y_>$ stands for the greater of y or y'. The functions $\overrightarrow{f}(y)$ and $\overleftarrow{f}(y)$ are two linearly independent solutions of the homogeneous form of (D-41): $\overleftarrow{f}(y)$ satisfies the boundary condition for y<y' (i.e., at y=-D) while $\overrightarrow{f(y)}$ satisfies the boundary condition for

$y > y'$ (i.e., at $y=0$). One finds that

$$\overrightarrow{f}(y) = e^{-i\kappa^- y} - \overrightarrow{r}_1(0) e^{i\kappa^- y} \quad , \tag{D-45}$$

with

$$\overrightarrow{r}_1(0) = \frac{Z_1 - Z_1^-}{Z_1 + Z_1^-} \quad . \tag{D-46}$$

Similarly,

$$\overleftarrow{f}(y) = \overleftarrow{r}_1 e^{-i\kappa^-(y+D)} - e^{i\kappa^-(y+D)} \quad . \tag{D-47}$$

The constant A in (D-44) is determined by first integrating both sides of (D-47) with respect to y between the limits $y = y' - \varepsilon \equiv y'_-$ and $y = y' + \varepsilon \equiv y'_+$ and requiring that $G_1(y,y')$ be continuous at $y=y'$. This yields

$$\dot{G}_1(y'_+, y') - \dot{G}_1(y'_-, y') = i\kappa^- Y_1^- \quad , \tag{D-48}$$

where the dot denotes differentiation with respect to the first of the two variables forming the arguments of G_1. Since

$$\dot{G}_1(y'_+, y') = A\, \overleftarrow{f}(y') \dot{\overrightarrow{f}}(y'_+) \quad , \tag{D-49}$$

$$\dot{G}_1(y'_-, y') = A\, \dot{\overleftarrow{f}}(y'_-) \overrightarrow{f}(y') \quad , \tag{D-50}$$

one finds upon setting $y'_+ = y'_- = y'$ and substituting in (D-48) that

$$A = \frac{i\kappa^{-} Y_1^{-}}{\dot{\overrightarrow{f}}(y')\overleftarrow{f}(y') - \overrightarrow{f}(y')\dot{\overleftarrow{f}}(y')} \quad . \tag{D-51}$$

Employing (D-47) and (D-45), a straightforward calculation yields

$$A = \frac{1}{2} \quad \frac{Y_1^{-}}{e^{i\kappa^{-}D} - \overleftarrow{\gamma}_1 \overrightarrow{\Gamma}_1(0)\, e^{-i\kappa^{-}D}} \quad . \tag{D-52}$$

Since (D-43) must reduce to (D-44) at $y=0$ the unknown function $T(y')$ in (D-43) is given by

$$T(y') = A\, \overleftarrow{f}(y')\, \overrightarrow{f}(0) \quad ,$$

whence the complete expression for $G_1(y,y')$ in (D-43) becomes

$$G_1(y,y') = \frac{Y_1^{-}}{2} \; \frac{\overleftarrow{\gamma}_1\, e^{-i\kappa^{-}(y'+D)} - e^{i\kappa^{-}(y'+D)}}{e^{i\kappa^{-}D} - \overleftarrow{\gamma}_1 \overrightarrow{\Gamma}_1(0)\, e^{i\kappa^{-}D}} \left[1-\overrightarrow{\Gamma}_1(0)\right] e^{-i\kappa y} \quad . \tag{D-53}$$

Also, from (37),

$$V_1(y,y') = \frac{Y_1^{-}}{2Y_1} \; \frac{\gamma_1\, e^{-i\kappa^{-}(y'+D)} - e^{-i\kappa^{-}(y'+D)}}{e^{i\kappa^{-}D} - \overleftarrow{\gamma}_1 \overrightarrow{\Gamma}_1(0)\, e^{-i\kappa^{-}D}} \left[1-\overrightarrow{\Gamma}_1(0)\right] e^{-i\kappa y} . \tag{D-54}$$

<u>Canonical Problem 2</u>. (E-mode voltage excited by a unit current source)

$$
-\frac{d\hat{G}_1}{dy} = i\kappa^- Z_1^- I_1 \quad ,
$$

$$
-\frac{dI_1}{dy} = i\kappa^- Y_1^- \hat{G}_1 + \delta(y-y') \quad ; \quad y \leq 0 \quad , \tag{D-55}
$$

$$
-\frac{d\hat{G}_1}{dy} = i\kappa Z_1 I_1 \quad ,
$$

$$
-\frac{dI_1}{dy} = i\kappa Y_1 \hat{G}_1 \quad ; \; y \geq 0 \quad . \tag{D-56}
$$

The second order differential equation for $\hat{G}_1$ reads

$$
\frac{d^2}{dy^2}\hat{G}_1 + (\kappa^-)^2 \hat{G}_1 = i\kappa^- Z_1^- \, \delta(y-y') \; ; \; y < 0 \quad , \tag{D-57}
$$

$$
\frac{d^2}{dy^2}\hat{G}_1 + \kappa^2 \hat{G}_1 = 0 \quad ; \quad y > 0 \quad . \tag{D-58}
$$

The solution can be written at once by comparison with Canonical Problem 1. Thus, comparing (D-41) with (D-57) one observes that the solution of (D-57) is again given by (D-53) provided Y_1^- is replaced by Z_1^- and the signs of $\hat{Y}_1$ and $\hat{I}_1(0)$ are reversed. Therefore, the solution to Canonical Problem 2 is

$$\hat{G}_1(y,y') = -\frac{Z_1^-}{2}\;\frac{\overleftarrow{\gamma}_1 e^{-i\kappa^-(y'+D)} + e^{i\kappa^-(y'+D)}}{e^{i\kappa^- D} - \overleftarrow{\gamma}_1\overrightarrow{\Gamma}_1(0)\, e^{-i\kappa^- D}}\left[1+\overrightarrow{\Gamma}_1(0)\right]e^{-i\kappa y}\;, \tag{D-59}$$

and

$$I_1(y,y') = \frac{Z_1^-}{2Z_1}\;\frac{\overleftarrow{\gamma}_1 e^{-i\kappa^-(y'+D)} + e^{i\kappa^-(y'+D)}}{e^{i\kappa^- D} - \overleftarrow{\gamma}_1\overrightarrow{\Gamma}_1(0)\, e^{-i\kappa^- D}}\left[1+\overrightarrow{\Gamma}_1(0)\right]e^{-i\kappa y}\;. \tag{D-60}$$

Finally, for the third cononical problem [corresponding to the H-mode voltage in (D-34)] one has to solve the differential equations

$$\frac{d^2\hat{G}_2}{dy^2} + (\kappa^-)^2\,\hat{G}_2 = i\kappa^- Z_2^-\,\delta(y-y') \quad ; \quad y < 0 \;, \tag{D-61}$$

$$\frac{d^2\hat{G}_2}{dy^2} + \kappa^2\hat{G}_2 = 0 \quad ; \quad y > 0 \;. \tag{D-62}$$

The solution follows immediately by comparison with (D-59):

$$\hat{G}_2(y,y') = -\frac{Z_1^-}{2}\;\frac{\overleftarrow{\gamma}_2 e^{-i\kappa^-(y'+D)} + e^{i\kappa^-(y'+D)}}{e^{i\kappa^- D} - \overleftarrow{\gamma}_2\overrightarrow{\Gamma}_2(0)\, e^{-i\kappa^- D}}\left[1+\overrightarrow{\Gamma}_2(0)\right]e^{-i\kappa y}\;. \tag{D-63}$$

Also, the corresponding current is

$$I_2(y,y') = -\frac{Z_2^-}{2Z_2}\;\frac{\overleftarrow{\gamma}_2 e^{-i\kappa^-(y'+D)} + e^{i\kappa^-(y'+D)}}{e^{i\kappa^- D} - \overleftarrow{\gamma}_2\overrightarrow{\Gamma}_2(0)\, e^{i\kappa^- D}}\left[1+\Gamma_2(0)\right]e^{-i\kappa y}\;, \tag{D-64}$$

where

$$\overrightarrow{\Gamma}_2(0) = \frac{Z_2 - Z_2^-}{Z_2 + Z_2^-} \quad . \tag{D-65}$$

As a notational convenience, let

$$T_{I1}(y',\omega,\underline{k}_T) = \frac{Y_1^-}{2}\,\frac{\overleftarrow{\Gamma}_1 e^{-i\kappa^-(y'+D)} - e^{i\kappa^-(y'+D)}}{e^{i\kappa^- D} - \overleftarrow{\Gamma}_1\overrightarrow{\Gamma}_1(0)e^{-i\kappa^- D}}\left[1-\overrightarrow{\Gamma}_1(0)\right] \quad , \tag{D-66}$$

$$T_{v1}(y',\omega,\underline{k}_T) = -\frac{Z_1^-}{2}\,\frac{\overleftarrow{\Gamma}_1 e^{-i\kappa^-(y'+D)} + e^{i\kappa^-(y'+D)}}{e^{i\kappa^- D} - \overleftarrow{\Gamma}_1\overrightarrow{\Gamma}_1(0)e^{-i\kappa^- D}}\left[1+\overrightarrow{\Gamma}_1(0)\right] \quad , \tag{D-67}$$

$$T_{v2}(y',\omega,\underline{k}_T) = -\frac{Z_2^-}{2}\,\frac{\overleftarrow{\Gamma}_2 e^{-i\kappa^-(y'+D)} + e^{i\kappa^-(y'+D)}}{e^{i\kappa^- D} - \overleftarrow{\Gamma}_2\overrightarrow{\Gamma}_2(0)e^{-i\kappa^- D}}\left[1+\overrightarrow{\Gamma}_2(0)\right] \quad . \tag{D-68}$$

The solutions to the three cononical problems may now be written

$$G_1(y,y') = T_{I1}(y',\omega,\underline{k}_T)\, e^{-i\kappa y} \quad , \tag{D-69a}$$

$$V_1(y,y') = Z_1 T_{I1}(y'.\omega,\underline{k}_T)\, e^{-i\kappa y} \quad , \tag{D-69b}$$

$$\hat{G}_1\,(y,y') = T_{v1}(y',\omega,\underline{k}_T)\, e^{-i\kappa y} \quad , \tag{D-70a}$$

$$I_1(y,y') = Y_1 T_{v1}(y',\omega,\underline{k}_T)\, e^{-i\kappa y} \quad , \tag{D-70b}$$

$$\hat{G}_2(y,y') = T_{v2}(y',\omega,\underline{k}_T)\, e^{-i\kappa y} \quad , \tag{D-71a}$$

$$I_2(y,y') = Y_2 T_{v2}(y',\omega,\underline{k}_T)\, e^{-i\kappa y} \quad . \tag{D-71b}$$

These quantities represent the solutions of (D-21) and (D-22) when the excitations $\zeta(y)$, $-\eta(y)$ and $\xi(y)$ in (D-33) and (D-34) are replaced by delta functions. Consequently, employing the principle of superposition, the solution of (D-21) and (D-22) for excitations ξ, η, ζ are given by

$$V_1^+(y) = e^{-i\kappa y}\left[Z_1 \int_{-D}^{0} T_{I1}(y')\zeta(y')dy' - \int_{-D}^{0} T_{v1}(y')\eta(y')dy'\right], \tag{D-72a}$$

$$I_1^+(y) = e^{-i\kappa y}\left[\int_{-D}^{0} T_{I1}(y')\zeta(y')dy' - Y_1 \int_{-D}^{0} T_{v1}(y')\eta(y')dy'\right], \tag{D-72b}$$

$$V_2^+(y) = e^{-i\kappa y}\left[\int_{-D}^{0} T_{v2}(y')\xi(y')dy'\right] , \tag{D-73a}$$

$$I_2^+(y) = e^{-i\kappa y}\left[Y_2 \int_{-D}^{0} T_{v2}(y')\xi(y')dy'\right] . \tag{D-73b}$$

The components of $\underline{E}$, $\underline{H}$ follow from (D-19) and (D-11):

$$E_z^+ = \frac{k_z}{k_T} V_1^+(y) + \frac{k_x}{k_T} V_2^+(y) , \tag{D-74a}$$

$$E_x^+ = \frac{k_x}{k_T} V_1^+(y) - \frac{k_z}{k_T} V_2^+(y) , \tag{D-74b}$$

$$E_y^+ = -\frac{k_T}{\omega\varepsilon_o} I_1^+(y) , \tag{D-74c}$$

$$H_z^+ = -\frac{k_x}{k_T} I_1^+(y) + \frac{k_z}{k_T} I_2^+(y) \quad , \qquad \text{(D-75a)}$$

$$H_x^+ = \frac{k_z}{k_T} I_1^+(y) + \frac{k_x}{k_T} I_2^+(y) \quad , \qquad \text{(D-75b)}$$

$$H_y^+ = -\frac{k_T}{\omega\mu_o} \; V_2^+(y) \quad . \qquad \text{(D-75c)}$$

Equations (D-72) and (D-73) when substituted in (D-74) and (D-75) give transformed electromagnetic field quantities that enter into the representations (D-3). To make the dependence of the excitation coefficients on the fluid velocity explicit, one can write

$$\xi(\underline{k}_T,\omega,y') = \sigma\underline{a}(\underline{k}_T) \cdot \underline{T}(\underline{k}_T,\omega,y') \quad , \qquad \text{(D-76a)}$$

$$\eta(\underline{k}_T,\omega,y') = \sigma\underline{b}(\underline{k}_T) \cdot \underline{T}(\underline{k}_T,\omega,y') \quad , \qquad \text{(D-76b)}$$

$$\zeta(\underline{k}_T,\omega,y') = -ik_T\underline{c} \cdot \underline{T}(\underline{k}_T,\omega,y') \quad , \qquad \text{(D-76c)}$$

where

$$\underline{a}(\underline{k}_T) = \frac{\underline{k}_T}{k_T} B_{oy} - \left(\underline{B}_o \cdot \frac{\underline{k}_T}{k_T}\right)\underline{y}_o \quad , \qquad \text{(D-77a)}$$

$$\underline{b}(\underline{k}_T) = \left(\frac{\underline{k}_T}{k_T} \times \underline{y}_o\right) B_{oy} - \left[\underline{B}_o \cdot \left(\frac{\underline{k}_T}{k_T} \times \underline{y}_o\right)\right] \underline{y}_o \quad , \qquad \text{(D-77b)}$$

$$\underline{c} = \underline{B}_o \times \underline{y}_o \quad . \qquad \text{(D-77c)}$$

The cartesian components of the auxiliary vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are

$$a_z = \frac{k_z}{k_T} B_{oy} , \quad a_x = \frac{k_x}{k_T} B_{oy} , \quad a_y = -\underline{B}_o \cdot \frac{\underline{k}_T}{k_T} ,$$

$$b_z = \frac{k_x}{k_T} B_{oy} , \quad b_x = -\frac{k_z}{k_T} B_{oy} , \quad b_y = -\underline{B}_o \left(\frac{\underline{k}_T}{k_T} \times \underline{y}_o\right) ,$$

$$c_z = B_{ox} , \quad c_x = -B_{oz} , \quad c_y = 0 .$$

The source terms in (D-76) ξ, η, ζ are now substituted in (D-72) and (D-73) to obtain

$$V_1^+(y) = e^{-i\kappa y} \int_{-D}^{0} dy' [(-ik_T Z_1 T_{I1} c_z - T_{v1} \sigma b_z)\tau_z \qquad \text{(D-78)}$$

$$+ (-ik_T Z_1 T_{I1} c_x - T_{v1} \sigma b_x)\tau_x$$

$$-(T_{v1} \sigma b_y)\tau_y] ,$$

$$V_2^+(y) = e^{-i\kappa y} \int_{-D}^{0} dy' \, [\sigma T_{v2} a_z \tau_z + \sigma T_{v2} a_x \tau_x + \sigma T_{v2} a_y \tau_y] , \qquad \text{(D-79)}$$

$$I_1^+(y) = e^{-i\kappa y} \int_{-D}^{0} dy' \, [(-ik_T c_z T_{I1} - Y_1 T_{v1} \sigma b_z)\tau_z$$

$$+ (-ik_T c_x T_{I1} - Y_1 T_{v1} \sigma b_x)\tau_x + (-Y_1 T_{v1} \sigma b_y)\tau_y] , \qquad \text{(D-80)}$$

$$I_2^+(y) = e^{-i\kappa y} \int_{-D}^{0} dy' \, [Y_2 T_{v2} \sigma a_z \tau_z + Y_2 T_{v2} \sigma a_x \tau_x \qquad \text{(D-81)}$$

$$+ Y_2 T_{v2} \sigma a_y \tau_y] .$$

The spatial Fourier transforms of the electric and magnetic fields can now be obtained by substituting (D-78-D-81) into (D-74) and (D-75). The final results are best expressed in matrix form with

$$T(\underline{k}_T,\omega,y') = \begin{bmatrix} T_z(\underline{k}_T,\omega,y') \\ T_x(\underline{k}_T,\omega,y') \\ T_y(\underline{k}_T,\omega,y') \end{bmatrix} , \tag{D-82}$$

$$\underline{H}^+(\underline{k}_T,\omega,y) = e^{-i\kappa y} \int_{-D}^{0} dy' \;\; \underline{\underline{S}}^{(H)}(\underline{k}_T,\omega,y')\underline{T}(\underline{k}_T,\omega,y') \;\; , \tag{D-83}$$

$$\underline{E}^+(\underline{k}_T,\omega,y) = e^{-i\kappa y} \int_{-D}^{0} dy' \;\; \underline{\underline{S}}^{(E)}(\underline{k}_T,\omega,y')\underline{T}(\underline{k}_T,\omega,y') \;\; . \tag{D-84}$$

The elements of the 3 x 3 matrices $\underline{\underline{S}}^{(H)}$ and $\underline{\underline{S}}^{(E)}$ are:

$$S_{zz}^{(H)} = B_{ox} i k_x T_{I1} + B_{oy}\left[Y_1 T_{v1} k_x^2 + Y_2 T_{v2} k_z^2\right] \frac{\sigma}{k_T^2} \;, \tag{D-84a}$$

$$S_{zx}^{(H)} = -B_{oz} i k_x T_{I1} + B_{oy}\left[-Y_1 T_{v1} k_z k_x + Y_2 T_{v2} k_z k_x\right] \frac{\sigma}{k_T^2} \;, \tag{D-85b}$$

$$S_{zy}^{(H)} = B_{oz}\left[-\frac{k_x^2\sigma}{k_T^2} Y_1 T_{v1} - \frac{k_z^2\sigma}{k_T^2} Y_2 T_{v2}\right] \tag{D-85c}$$

$$+B_{ox}\left[\frac{k_z k_x \sigma}{k_T^2} Y_1 T_{v1} - \frac{k_z k_x \sigma}{k_T^2} Y_2 T_{v2}\right] ,$$

$$S_{xz}^{(H)} = - B_{ox}\, ik_z\, T_{I1} + B_{oy}\,[-Y_1\, T_{v1} + Y_2\, T_{v2}]\,\frac{\sigma k_z k_x}{k_T^2}\quad , \qquad \text{(D-85d)}$$

$$S_{xx}^{(H)} = B_{oz}\, ik_z\, T_{I1} + B_{oy}\,[Y_1\, T_{v1}\, k_z^2 + Y_2\, T_{v2}\, k_x^2]\,\frac{\sigma}{k_T^2}\quad , \qquad \text{(D-85e)}$$

$$S_{xy}^{(H)} = B_{oz}[Y_1\, T_{v1} - Y_2\, T_{v2}]\,\frac{\sigma k_z k_x}{k_T^2} - B_{ox}\,[Y_1\, T_{v1}\, k_z^2 + Y_2\, T_{v2}\, k_x^2]\,\frac{\sigma}{k_T^2}\, , \qquad \text{(D-85f)}$$

$$S_{yz}^{(H)} = B_{oy}\left[-\frac{T_{v2}\, k_z\, \sigma}{\omega\mu_o}\right]\quad , \qquad \text{(D-85g)}$$

$$S_{yx}^{(H)} = B_{oy}\left[-\frac{T_{v2}\, k_x\, \sigma}{\omega\mu_o}\right]\quad , \qquad \text{(D-85h)}$$

$$S_{yy}^{(H)} = B_{oz}\left[T_{v2}\,\frac{\sigma k_z}{\omega\mu_o}\right] + B_{ox}\left[T_{v2}\,\frac{\sigma k_x}{\omega\mu_o}\right]\quad , \qquad \text{(D-85i)}$$

$$S_{zz}^{(E)} = B_{ox}\,[-ik_z\, Z_1\, T_{I1}] + B_{oy}\,[-T_{v1} + T_{v2}]\,\frac{\sigma k_z k_x}{k_T^2}\quad , \qquad \text{(D-86a)}$$

$$S_{zx}^{(E)} = B_{oz}\,[ik_z\, Z_1\, T_{I1}] + B_{oy}\,[k_z^2\, T_{v1} + k_x^2\, T_{v2}]\,\frac{\sigma}{k_T^2}\quad , \qquad \text{(D-86b)}$$

$$S_{zy}^{(E)} = B_{oz}\,[T_{v1} - T_{v2}]\,\frac{k_z k_x \sigma}{k_T^2} + B_{ox}\,[-T_{v1}\, k_z^2 - T_{v2}\, k_x^2 - T_{v2}\, k_x^2]\,\frac{\sigma}{k_T^2}\quad , \qquad \text{(D-86c)}$$

$$S_{xz}^{(E)} = B_{ox}\,[-ik_x\, Z_1\, I_{T1}] + B_{oy}\,[-T_{v1}\, k_x^2 - T_{v2}\, k_z^2]\,\frac{\sigma}{k_T^2}\quad , \qquad \text{(D-86d)}$$

$$S_{xx}^{(E)} = B_{oz}\,[ik_x\, Z_1\, I_{T1}] + B_{oy}\,[T_{v1} - T_{v2}]\,\frac{\sigma k_z k_x}{k_T^2}\quad , \qquad \text{(D-86e)}$$

$$S_{xy}^{(E)} = B_{oz}\,[T_{v1}\,k_x^2 + T_{v2}\,k_z^2]\frac{\sigma}{k_T^2} + B_{ox}\,[-T_{v1} + T_{v2}]\frac{\sigma k_z k_x}{k_T^2} \quad , \tag{D-86f}$$

$$S_{yz}^{(E)} = B_{ox}\left[\frac{ik_T^2 T_{I1}}{\omega\varepsilon_o}\right] + B_{oy}\,[Y_1 T_{v1}\,k_x\frac{\sigma}{\omega\varepsilon_o}] \quad , \tag{D-86g}$$

$$S_{yx}^{(E)} = -B_{oz}\left[\frac{ik_T^2 T_{I1}}{\omega\varepsilon_o}\right] + B_{oy}\left[-Y_1\,T_{v1}\,\frac{\sigma k_z}{\omega\varepsilon_o}\right] \quad , \tag{D-86h}$$

$$S_{yy}^{(E)} = B_{oz}\left[-Y_1\,T_{v_1}\,\frac{\sigma k_x}{\omega\varepsilon_o}\right] + B_{ox}\left[Y_1\,T_{v1}\,\frac{\sigma k_z}{\omega\varepsilon_o}\right] \quad . \tag{D-86i}$$

The matrices $\underline{\underline{S}}^{(H)}$ and $\underline{\underline{S}}^{(E)}$ are not independent since (D-83) and (D-84) must satisfy the homogeneous linear equations (D-5a) and (D-5b). One finds directly from (D-85) and (D-86) that

$$(\underline{k}_T - \underline{y}_o\kappa) \times \underline{\underline{S}}^{(H)} = -\omega\varepsilon_o\underline{\underline{S}}^{(E)} \quad , \tag{D-87a}$$

or

$$\begin{bmatrix} 0 & -\kappa & k_x \\ \kappa & 0 & -k_z \\ -k_x & k_z & 0 \end{bmatrix} \underline{\underline{S}}^{(H)} = -\omega\varepsilon_o\underline{\underline{S}}^{(E)} \quad ; y > 0 \quad . \tag{D-87b}$$

The temporal Fourier transforms of the fields are obtained by combining (D-83)(D-84) with (D-3a) and (D-3b) to obtain

$$\tilde{\underline{E}}(\underline{r},\omega) = \iint_{-\infty}^{\infty} d^2 k_T\; e^{-i\underline{k}_T\cdot\underline{\rho}-i\kappa y} \int_{-D}^{0} dy'\; \underline{\underline{S}}^{(H)}(\underline{k}_T,\omega,y')\underline{T}(\underline{k}_T,\omega,y') \quad , \tag{D-88}$$

$$\underline{\tilde{H}}(\underline{r},\omega) = \iint_{-\infty}^{\infty} d^2\underline{k}_T\, e^{-i\underline{k}_T\cdot\underline{\rho}-i\kappa y} \int_{-D}^{0} dy'\, \underline{\underline{S}}^{(E)}(\underline{k}_T,\omega,y')\underline{T}(\underline{k}_T,\omega,y') \quad . \qquad \text{(D-89)}$$

Finally, the time varying fields $\underline{E}(\underline{r},t)$, $\underline{H}(\underline{r},t)$ are

$$\underline{E}(\underline{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \iint_{-\infty}^{\infty} d^2\underline{k}_T\, e^{-i\underline{k}_T\cdot\underline{\rho}-i\kappa y} \int_{-D}^{0} dy'\, \underline{\underline{S}}^{(E)}(\underline{k}_T,\omega,y')\, \underline{T}(\underline{k}_T,\omega,y') \;, \qquad \text{(D-90)}$$

$$\underline{H}(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \iint_{-\infty}^{\infty} d^2\underline{k}_T\, e^{-i\underline{k}_T\cdot\underline{\rho}-i\kappa y} \int_{-D}^{0} dy'\, \underline{\underline{S}}^{(H)}(\underline{k}_T,\omega,y')\, \underline{T}(\underline{k}_T,\omega,y') \;. \qquad \text{(D-91)}$$

Alternatively, (D-90) and (D-91) may be written as

$$\underline{E}(\underline{r},t) = \iint_{-\infty}^{\infty} \hat{\underline{E}}(\underline{k}_T,y,t)\, e^{-i\underline{k}_T\cdot\underline{\rho}}\, d^2\underline{k}_T \quad , \qquad \text{(D-92a)}$$

$$\underline{H}(\underline{r},t) = \iint_{-\infty}^{\infty} \hat{\underline{H}}(\underline{k}_T,y,t)\, e^{-i\underline{k}_T\cdot\underline{\rho}}\, d^2\underline{k}_T \quad , \qquad \text{(D-92b)}$$

with

$$\hat{\underline{E}}(\underline{k}_T,y,t) = \int_{-D}^{0} dy'\, \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\, e^{-i\kappa y}\, \underline{\underline{S}}^{(E)}(\underline{k}_T,\omega,y')\, \underline{T}(\underline{k}_T,\omega,y') \;, \qquad \text{(D-93a)}$$

$$\hat{\underline{H}}(\underline{k}_T,y,t) = \int_{-D}^{0} dy'\, \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\, e^{-i\kappa y}\, \underline{\underline{S}}^{(H)}(\underline{k}_T,\omega,y')\, \underline{T}(\underline{k}_T,\omega,y') \;. \qquad \text{(D-93b)}$$

Comparing the last two expressions with the results obtained under the quasi-static approximation in Chapter IV, Eq. (96), one can make the following identification:

$$\hat{\underline{G}}(\underline{k}_T;y,y') = \mu_0 \, e^{-k_T y} \, \underline{\underline{S}}^{(H)}(\underline{k}_T,0,y') \quad , \tag{D-94}$$

$$\hat{\underline{v}}(\underline{k}_T,y',t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \, d\omega \, \underline{T}(\underline{k}_T,\omega,y') \quad , \tag{D-95}$$

i.e., in the quasi-static approximation, the matrix $e^{-i\kappa y}\underline{\underline{S}}^{(H)}(\underline{k}_T,\omega,y)$ is approximated by its value at $\omega = 0$. One can show that with $\omega = 0$ use of Eq. (D-85) yields matrix elements consistent with Eqs. (98-100) in Chapter IV.

We now apply (D-90) and (D-91) to compute the electromagnetic fields induced by small amplitude surface waves. For simplicity consider a unidirectional surface wave with frequency Ω, propagation constant $\underline{K}$, and $\Omega^2 = Kg$ (deep ocean). In Eq. (A-22), Appendix A, we set $A^+(\underline{k}_T) = A\,\delta(\underline{k}_T - \underline{K})$, $A^-(\underline{k}_T) = 0$, so that the velocity potential is

$$\phi(\underline{\rho},y,t) = i\Omega K^{-1} A \, e^{Ky} \, e^{-i\underline{K}\cdot\underline{\rho} + i\Omega t} \quad .$$

The temporal and spatial Fourier transform of the velocity field is

$$\underline{T}(\underline{k}_T,y,\omega) = 2\pi A K^{-1}\Omega(\underline{y}_0 K - i\underline{K}) \, e^{Ky} \delta(\underline{k}_T - \underline{K})\delta(\omega - \Omega) \quad . \tag{D-96}$$

With (D-96) substituted into (D-90) and (D-91) the expressions for the electromagnetic fields for all $y \geq 0$ become

$$\underline{E}(\underline{r},t) = iAK^{-1}\Omega \, e^{-i\underline{K}\cdot\underline{\rho} - i\kappa y + i\Omega t} \int_{-\infty}^{0} e^{Ky'} \underline{\underline{S}}^{(E)}(\underline{K},\Omega,y')\cdot(\underline{y}_0 K - i\underline{K}) dy' \, , \tag{D-97}$$

$$\underline{H}(\underline{r},t) = iAK^{-1}\Omega \, e^{-i\underline{K}\cdot\underline{\rho} - i\kappa y + i\Omega t} \int_{-\infty}^{0} e^{Ky'} \underline{\underline{S}}^{(H)}(\underline{K},\Omega,y')\cdot(\underline{y}_0 K - i\underline{K}) \, dy' \, . \tag{D-98}$$

The integrals are taken over the semi-infinite y domain, in consonance with the assumption of a "deep" ocean. These integrals yield spatially independent vectors that depend on $\underline{K}$ (and hence frequency), and the direction and magnitude of the earth's magnetic field. The induced electromagnetic fields have the same time-harmonic dependence as the hydrodynamic surface wave, and also exhibit a propagation wave character in the transverse (x,z) plane. The dependence on y above the air-water interface of the form exp$-i\kappa y$ is actually an attenuation, since

$$\kappa = \sqrt{\left(\frac{\Omega}{c}\right)^2 - K^2} \tag{D-99}$$

with c the speed of light in vacuo. From the dispersion relationship

$$\kappa = \sqrt{\frac{\Omega^2}{c^2} - \frac{\Omega^4}{g^2}} = K\sqrt{\frac{g^2}{\Omega^2 c^2} - 1} \tag{D-100}$$

$$= K\sqrt{\left(\frac{V_p}{c}\right)^2 - 1} \quad ,$$

where $V_p = g/\Omega$ is the phase velocity of the surface water wave. Since $V_p/c \ll 1$ one has $\kappa \cong \pm iK$ so that exp $- i\kappa y \cong$ exp $\pm Ky$. On physical grounds, only the negative sign applies, and (D-97) and (D-98) yield fields that decay exponentially with increasing height above the ocean.

To simplify matters, let the earth's magnetic field lie entirely in the xy plane. The geometrical relationship between the surface wave propagation vector $\underline{K}$ and $\underline{B}_o$,

$$\underline{B}_o = B_{ox}\underline{x}_o + B_{oy}\underline{y}_o \quad , \tag{D-101}$$

is shown in Figs. 2, 3, Chapter V, page 60 (where the notation $\underline{k}_T$ is employed for $\underline{K}$). The propagation vector of the surface wave will be represented in polar form $K_x = K \cos w$, $K_z = K \sin w$, so

that with $\alpha = w$ in Fig. 2 the unit vector $\underline{\ell}_1$ points in the direction of propagation of the surface wave.

We consider first the magnetic field and let

$$\underline{s}^{(H)} = \underline{\underline{S}}^{(H)}(\underline{k},\Omega,y') \cdot (\underline{y}_0 K - i\underline{K}) = \underline{\underline{S}}^{H} \begin{bmatrix} -iK_z \\ -iK_x \\ K \end{bmatrix} = \begin{bmatrix} s_z^{(H)} \\ s_x^{(H)} \\ s_y^{(H)} \end{bmatrix} . \tag{D-101}$$

With the aid of (D-85) one finds

$$s_z^{(H)} = B_{ox} K_z K_x [T_{I1} + \frac{\sigma}{K} (Y_1 T_{v1} - Y_2 T_{v2})] - i\sigma K_z Y_2 T_{v2} B_{oy} , \tag{D-102a}$$

$$s_x^{(H)} = -B_{ox} [K_z^2 T_{I1} + \frac{\sigma}{K} (Y_1 T_{v1} K_z^2 + Y_2 T_{v2} K_x^2)] - i\sigma K_x Y_2 T_{v2} B_{oy} , \tag{D-102b}$$

$$s_y^{(H)} = B_{ox} \frac{\sigma K_x K}{\Omega \mu_o} T_{v2} + i\sigma \frac{K^2}{\Omega \mu_o} T_{v2} B_{oy} . \tag{D-102c}$$

In an infinitely deep ocean one can set $\overleftarrow{Y}_1$ and $\overleftarrow{Y}_2$ to zero in (D-66) through (D-68) to obtain

$$T_{I1} = - \frac{Y_1^-}{2} [1-\overrightarrow{\Gamma}_1(0)] e^{i\kappa^- y'} = - \frac{e^{i\kappa^- y'}}{Z_1+Z_1^-} , \tag{D-103a}$$

$$T_{v1} = - \frac{Z_1^-}{2} [1+\overrightarrow{\Gamma}_1(0)] e^{i\kappa^- y'} = - \frac{e^{i\kappa^- y'}}{Y_1+Y_1^-} , \tag{D-103b}$$

$$T_{v2} = - \frac{Z_2^-}{2} [1+\overrightarrow{\Gamma}_2(0)] e^{i\kappa^- y'} = - \frac{e^{i\kappa^- y'}}{Y_2+Y_2^-} . \tag{D-103c}$$

For the electric field one has

$$\underline{s}^{(E)} = \underline{\underline{S}}^{(E)}(\underline{K},\Omega,y')\cdot(\underline{y}_oK-i\underline{K}) = \underline{\underline{S}}^{(E)}\begin{bmatrix}-iK_z\\-iK_x\\K\end{bmatrix} = \begin{bmatrix}s_z^{(E)}\\s_x^{(E)}\\s_y^{(E)}\end{bmatrix} \quad . \tag{D-104}$$

Employing (86),

$$s_z^{(E)} = -B_{ox}[K_z^2Z_1T_{I1} + \frac{\sigma}{K}(T_{v1}K_z^2 + T_{v2}K_x^2)] - i\sigma K_xT_{v2}B_{oy} \quad , \tag{D-105a}$$

$$s_x^{(E)} = -B_{ox}[Z_1T_{I1} + \frac{\sigma}{K}(T_{v1}-T_{v2})]\,K_zK_x + i\sigma K_zT_{v2}B_{oy} \quad , \tag{D-105b}$$

$$s_y^{(E)} = B_{ox}\frac{KK_z}{\Omega\epsilon_o}[KT_{I1} + \sigma Y_1T_{v1}] \quad . \tag{D-105c}$$

The interpretation of (D-102) and (D-105) is facilitated by resolving the components parallel to the xz plane along $\underline{K}$ and a direction normal to $\underline{K}$. The new vectors labeled $\underline{s}_1^{(H)}$, $\underline{s}_3^{(H)}$, $\underline{s}_1^{(E)}$, $\underline{s}_3^{(E)}$ are

$$\underline{s}^{(H)} = \underline{s}_1^{(H)} + \underline{s}_3^{(H)} + \underline{y}_o\, s_y^{(H)} \quad ,$$

$$\underline{s}^{(E)} = \underline{s}_1^{(E)} + \underline{s}_3^{(E)} + \underline{y}_o\, s_y^{(E)} \quad ,$$

$$
\begin{bmatrix} s_z^{(E,H)} \\ s_x^{(E,H)} \\ s_y^{(E,H)} \end{bmatrix} = \begin{bmatrix} \sin w & \cos w & 0 \\ \cos w & -\sin w & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1^{(E,H)} \\ s_3^{(E,H)} \\ s_y^{(E,H)} \end{bmatrix} . \qquad \text{(D-106)}
$$

With the aid of (D-102) and (D-105) one finds

$$s_1^{(H)} = -B_{ox}\sigma Y_2 T_{v2} K_x - B_{oy} i\sigma K Y_2 T_{v2} , \qquad \text{(D-107a)}$$

$$s_3^{(H)} = B_{oy} K_z [K T_{I1} + \sigma Y_1 T_{v1}] , \qquad \text{(D-107b)}$$

$$s_1^{(E)} = -B_{ox} K_z [K Z_1 T_{I1} + \sigma T_{v1}] , \qquad \text{(D-107c)}$$

$$s_3^{(E)} = -B_{ox} \sigma K_x T_{v2} - i\sigma K T_{v2} B_{oy} . \qquad \text{(D-107d)}$$

These relations, together with $s_y^{(H)}$, $s_y^{(E)}$ in (D-102c) and (D-105c), respectively, permit a decomposition of the electromagnetic fields into two surface wave modes: an E-mode having only an electric field in the direction of propogation and defined by the triplet $(s_y^{(E)}, s_3^{(H)}, s_1^{(E)})$, and an H-mode characterized by having only a magnetic field in the direction of propagation, with the triplet $(s_y^{(H)}, s_3^{(E)}, s_1^{(H)})$.

From (D-105c), (D-102c) and (D-107) one finds the ratios

$$Z^{(E)} = \frac{s_y^{(E)}}{s_3^{(H)}} = \frac{K}{\Omega\varepsilon_o} \quad , \tag{D-108a}$$

$$Z^{(H)} = -\frac{s_3^{(E)}}{s_y^{(H)}} = \frac{\Omega\mu_o}{K} \quad , \tag{D-108b}$$

which are recognized as the E-mode and H-mode characteristic impedance, respectively. Recalling that $V_p = \Omega/K$ is the phase velocity of the hydrodynamic surface wave, the characteristic impedances are also given by

$$Z^{(E)} = \zeta_o \frac{c}{V_p} \quad , \tag{D-109a}$$

$$Z^{(H)} = \zeta_o \frac{V_p}{c} \quad . \tag{D-109b}$$

where $c = 1/\sqrt{\mu_o\varepsilon_o}$ is the speed of light vacuo, and $\zeta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} =$ 377 ohms is the characteristic impedance of a place electromagnetic wave in free space.

It remains now to compute the field amplitudes in (D-97) and (D-98). The results are

$$\int_{-\infty}^{0} e^{Ky'} s_1^{(H)}(y')dy' = \sigma(B_{ox}\cos w + iB_{oy}) \frac{1}{(1+\sqrt{1+iq})^2} \quad , \tag{D-110a}$$

$$\int_{-\infty}^{0} e^{Ky'} s_3^{(H)}(y')dy' = -i\Omega\varepsilon_o \sin w \; B_{ox} \frac{1}{1+\frac{i\Omega\varepsilon_o}{\sigma}\sqrt{1+iq}} \quad , \tag{D-110b}$$

$$\int_{-\infty}^{0} e^{Ky'} s_y^{(H)}(y')dy' = -i\sigma(B_{ox}\cos w + iB_{oy}) \frac{1}{(1+\sqrt{1+iq})^2} , \qquad \text{(D-110c)}$$

$$\int_{-\infty}^{0} e^{Ky'} s_1^{(E)}(y')dy' = KB_{ox}\sin w \frac{1}{1+ \frac{i\Omega\varepsilon_o}{\sigma}\sqrt{1+iq}} , \qquad \text{(D-110d)}$$

$$\int_{-\infty}^{0} e^{Ky'} s_3^{(E)}(y')dy' = i\sigma\left(\frac{\Omega\mu_o}{K}\right)\cdot (B_{ox}\cos w + iB_{oy}) \frac{1}{(1+\sqrt{1+iq})^2} , \qquad \text{(D-110e)}$$

$$\int_{-\infty}^{0} e^{Ky'} s_y^{(E)}(y')dy' = -iKB_{ox}\sin w \frac{1}{1+ \frac{i\Omega\varepsilon_o}{\sigma}\sqrt{1+iq}} . \qquad \text{(D-110f)}$$

The dimensionless quantity q appearing in these equations is defined by

$$q = \frac{\Omega\mu_o\sigma}{K^2} . \qquad \text{(D-111)}$$

Thus far no approximations have been made, and (D-110) are "exact" to the extent that only the displacement current in the sea water has been neglected. First, one observes that the quantity

$$\frac{\Omega\varepsilon_o}{\sigma}\sqrt{1+iq} , \qquad \text{(D-112)}$$

appearing in (D-110b, d, f) can be set equal to zero since $\varepsilon_o \approx \frac{1}{36\pi} \times 10^{-9}$, $\sigma \sim 4$ mho/meter and $\Omega \sim 1$ (for surface waves). Second, with the aid of the dispersion relationship $K = \Omega^2 g$, (D-111) becomes

$$q \approx 4.827 \times 10^{-4}\Omega^{-3} .$$

For surface waves this quantity is usually small by comparison with unity. For example, for $\Omega = .169$ radians/sec = .0269 Hz, (which corresponds to a phase velocity of 36.4 meters/sec) $q \approx .1$.

Assuming $q \ll 1$ and setting to zero the quantity (D-112), the expressions for the electromagnetic fields take on a particularly simple form. Using the decomposition into E and H modes relative to the $\underline{K}$ direction, one finds:

a) E-mode fields -

$$E_y(x,y,z,t) = A\Omega B_{ox} \sin w \, e^{-Ky-i\underline{K}\cdot\underline{\rho}+i\Omega t} , \tag{D-113a}$$

$$H_3(x,y,z,t) = A\Omega \left(\frac{\Omega\varepsilon_o}{K}\right) B_{ox} \sin w \, e^{-Ky-i\underline{K}\cdot\rho+i\Omega t} , \tag{D-113b}$$

$$E_1(x,y,z,t) = iA\Omega B_{ox} \sin w \, e^{-Ky-i\underline{K}\cdot\underline{\rho}+i\Omega t} , \tag{D-113c}$$

b) H-mode fields -

$$H_y(x,y,z,t) = A \frac{\sigma\Omega}{4K} (B_{ox} \cos w + iB_{oy}) \, e^{-Ky-i\underline{K}\cdot\underline{\rho}+i\Omega t} , \tag{D-114a}$$

$$E_3(x,y,z,t) = -A \frac{\sigma\Omega}{4K} \left(\frac{\Omega\mu_o}{K}\right)\left(B_{ox} \cos w + iB_{oy}\right) e^{-Ky-i\underline{K}\cdot\rho+i\Omega t} , \tag{D-114b}$$

$$H_1(x,y,z,t) = iA \frac{\sigma\Omega}{4K} (B_{ox} \cos w + iB_{oy}) \, e^{-Ky-i\underline{K}\cdot\underline{\rho}+i\Omega t} . \tag{D-114c}$$

These are the fields of classic electromagnetic surface wave modes. They propagate without attenuation along the surface of the sea, their phase velocity and propagation direction being identical with that of the hydrodynamic surface wave. Electromagnetic surface waves are generally slow waves, i.e., having phase velocities less than the speed of light. The present instance affords an illustration of surface waves that are slow, indeed. It is of interest to observe that the electromagnetic surface waves above the ocean are in fact

indistinguishable from those that would arise in air above a dielectric interface for plane waves incident from within a dielectric half space and totally reflected at the interface. To obtain phase velocities as low as those of hydrodynamic surface waves, the refractive index must, of course, be extremely high. For example, if θ is the angle of incidence within the dielectric, then the product of $\sin\theta$ and the refractive index must be on the order of 10^7, which is probably well outside the range of dielectric constants attainable with existing materials. A discussion of the relationship between the fields in (D-113) and (D-114) and those given by the quasi-static approximation is presented in Chapter V-D.

APPENDIX E

SMALL-AMPLITUDE OCEAN INTERNAL WAVES*

Contents

*This appendix provides the background material on linear internal waves, which in other parts of this report is used in the analysis of induced magnetic fields.

I. LINEARIZED EQUATIONS FOR INTERNAL WAVES IN A PLANE STRATIFIED OCEAN

A. LINEAR INTERNAL WAVES IN THE ABSENCE OF MEAN SHEAR AND VISCOSITY

We choose a Cartesian coordinate system such that the xz plane coincides with the ocean surface and take $y \geq 0$ above the ocean surface. If one neglects Coriolis effects and viscosity, the hydrodynamic equations are

$$\rho \frac{\partial \underline{V}}{\partial t} + \rho \underline{V} \cdot \nabla \underline{V} + \nabla p + \rho g \underline{y}_o = 0 \ , \qquad \text{(E-1a)}$$

$$\nabla \cdot (\rho \underline{V}) = - \frac{\partial \rho}{\partial t} \ , \qquad \text{(E-1b)}$$

together with the incompressibility condition

$$\nabla \cdot \underline{V} = 0 \ , \qquad \text{(E-1c)}$$

where $\underline{V}$, ρ, p are the fluid velocity, density and pressure, respectively.

Internal waves are sustained by virtue of small fluctuations in density which in turn produce fluctuations in the gravitational forcing term. If we denote the mean density of ρ by $\overline{\rho}$, then

$$\rho = \overline{\rho} + \rho' \ ,$$

where $\overline{\rho'} = 0$ and the condition that the density fluctuations be small is then

$$\overline{(\rho')^2} \ll \overline{\rho}^2 \ .$$

Similarly, the pressure p is assumed to undergo small fluctuations about the mean $\bar{p}$, so that

$$p = \bar{p} + p' ,$$

where again $\overline{p'} = 0$.

In most theoretical work on internal waves it is assumed that the principal direct effect of small fluctuations in density is comprised in the gravitational restoring force. This fundamental assumption is referred to as the Boussinesq approximation. It entails the replacement of the density ρ appearing in the two inertia terms of the momentum equation by the mean density $\bar{\rho}$, while still retaining the fluctuating density component in the gravitational forcing term $\rho g \underline{y}_o$. Thus, subject to the Boussinesq approximation, (E-1a) is replaced by

$$\bar{\rho} \frac{\partial \underline{V}}{\partial t} + \bar{\rho}\, \underline{V} \cdot \nabla \underline{V} + \nabla p' + \nabla \bar{p} + \rho' g \underline{y}_o + \bar{\rho}\, g \underline{y}_o = 0 . \qquad \text{(E-2)}$$

The ocean is assumed horizontally stratified so that the mean density $\bar{\rho}$ is not a function of x and z. If in addition the mean density does not depend on time, the incompressibility condition (E-1c) together with the equation of continuity leads to the statement

$$\frac{\partial \rho'}{\partial t} + V_y \frac{d\, \bar{\rho}}{dy} = - \underline{V} \cdot \nabla \rho' , \qquad \text{(E-3)}$$

where V_y is the vertical component of fluid velocity.

As the next simplifying assumption we take the mean of all the fluid velocity components as zero, viz.,

$$\overline{\underline{V}} = 0 . \qquad \text{(E-4)}$$

The consequences of a nonzero mean velocity will be taken up at a later point. Upon carrying out a statistical averaging operation on (E-3) and taking account of (E-4) gives

$$\overline{\underline{V} \cdot \nabla\rho'} = 0 \; . \tag{E-5}$$

Similarly, the average of (E-2) yields

$$\bar{\rho}\; \overline{\underline{V} \cdot \nabla\underline{V}} + \nabla\bar{p} + \bar{\rho}\; g\underline{y}_o = 0 \; . \tag{E-6}$$

Employing this in (E-2) we have

$$\frac{\partial \underline{V}}{\partial t} + \frac{\nabla p'}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}}\, g\underline{y}_o = (\overline{\underline{V} \cdot \nabla\underline{V}} - \underline{V} \cdot \nabla\underline{V}) \; . \tag{E-7}$$

Equations (E-7), (E-3), and (E-1c) are the fundamental equations for a (zero mean) velocity field $\underline{V}$ induced by small fluctuations of density ρ' in a horizontally stratified ocean. The quantities

$$\alpha_V \equiv (\overline{\underline{V} \cdot \nabla\underline{V}} - \underline{V} \cdot \nabla\underline{V}) \; , \tag{E-8}$$

$$\beta_{\rho' V} = - \underline{V} \cdot \nabla\rho' \; , \tag{E-9}$$

appearing on the right of (E-7) and (E-3), respectively, are zero mean random functions (see E-5). If the (zero mean) fluctuating velocity field $\underline{V}$ is sufficiently small, the fluctuations of α_V, $\beta_{\rho' V}$ about their means are of a smaller order. With the aid of the usual statistical argument one can then approximate α_V, $\beta_{\rho' V}$ by their averages, viz.,

$$\alpha_V \approx \overline{\alpha_V} = 0 \; ,$$

$$\beta_{\rho' V} \approx \overline{\beta_{\rho' V}} = 0 \; .$$

Once this is done, the result is the set of homogeneous linear equations in $\underline{V}$, ρ' and p' that form the basis for the study of small amplitude internal waves:

$$\frac{\partial \underline{V}}{\partial t} + \frac{\nabla p'}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}} g\underline{y}_o = 0 \,, \qquad \text{(E-10a)}$$

$$\frac{\partial \rho'}{\partial t} + V_y \frac{d\bar{\rho}}{dy} = 0 \,. \qquad \text{(E-10b)}$$

The functions α_v and $\beta_{\rho',v}$ in (E-7) and (E-3) may be interpreted as "source" terms of a linear system of equations. If the variance of each of these source terms is of a smaller order than the variance of $\underline{V}$, an iteration procedure can be established whereby weakly nonlinear effects may be taken into account. For example, in the first iteration the source functions would be expressed in terms of the solution of the homogeneous linear system (E-10). In the next step one would solve the inhomogeneous linear system (E-7), (E-3) in which the sources would be expressed in terms of the $\underline{V}$ and ρ' determined in the preceding step. In the following, we shall concern ourselves only with the zeroth order linear system, viz., (E-10).

We now proceed to transform these equations into the wave equation for linear internal waves. Since there exist two slightly different versions of this wave equation in the published literature, we shall carry out the derivation in detail, thereby identifying the steps leading to the discrepancy.

Writing out (E-10a) in component form yields

$$\frac{\partial V_z}{\partial t} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} = 0 \,, \qquad \text{(E-11a)}$$

$$\frac{\partial V_x}{\partial t} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} = 0 \,, \qquad \text{(E-11b)}$$

$$\frac{\partial V_y}{\partial t} + \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial y} + \frac{\rho'}{\bar{\rho}} g = 0 \,. \qquad \text{(E-11c)}$$

After differentiating Eq. (E-11c) with respect to the horizontal coordinates x,z and Eq. (E-11a, b) with respect to y one obtains

$$\frac{\partial^2 V_y}{\partial t \partial z} + \frac{1}{\bar{\rho}} \frac{\partial^2 p'}{\partial z \partial y} + \frac{g}{\bar{\rho}} \frac{\partial \rho'}{\partial z} = 0 , \tag{E-12a}$$

$$\frac{\partial^2 V_y}{\partial t \partial x} + \frac{1}{\bar{\rho}} \frac{\partial^2 p'}{\partial x \partial y} + \frac{g}{\bar{\rho}} \frac{\partial \rho'}{\partial x} = 0 , \tag{E-12b}$$

$$\frac{\partial^2 V_z}{\partial t \partial y} + \frac{1}{\bar{\rho}} \frac{\partial^2 p'}{\partial z \partial y} + \frac{\partial p'}{\partial z} \frac{d}{dy} \left(\frac{1}{\bar{\rho}} \right) = 0 , \tag{E-12c}$$

$$\frac{\partial^2 V_x}{\partial t \partial y} + \frac{1}{\bar{\rho}} \frac{\partial^2 p'}{\partial x \partial y} + \frac{\partial p'}{\partial x} \frac{d}{dy} \left(\frac{1}{\bar{\rho}} \right) = 0 . \tag{E-12d}$$

We now eliminate the cross derivatives of p', and obtain the following two equations:

$$\frac{\partial}{\partial t} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \frac{\partial p'}{\partial z} \frac{d}{dy} \left(\frac{1}{\bar{\rho}} \right) - \frac{g}{\bar{\rho}} \frac{\partial \rho'}{\partial z} = 0 , \tag{E-13a}$$

$$\frac{\partial}{\partial t} \left(\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x} \right) + \frac{\partial p'}{\partial x} \frac{d}{dy} \left(\frac{1}{\bar{\rho}} \right) - \frac{g}{\bar{\rho}} \frac{\partial \rho'}{\partial x} = 0 . \tag{E-13b}$$

Substituting for $\frac{\partial p'}{\partial z}$ and $\frac{\partial p'}{\partial x}$ from (E-11), yields

$$\frac{\partial}{\partial t}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \bar{\rho}\,\frac{d}{dy}\left(\frac{1}{\bar{\rho}}\right)\frac{\partial V_z}{\partial t} - \frac{g}{\bar{\rho}}\,\frac{\partial \rho'}{\partial z} = 0\ , \qquad \text{(E-13c)}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}\right) - \bar{\rho}\,\frac{d}{dy}\left(\frac{1}{\bar{\rho}}\right)\frac{\partial V_x}{\partial t} - \frac{g}{\bar{\rho}}\,\frac{\partial \rho'}{\partial x} = 0\ . \qquad \text{(E-13d)}$$

Now use is made of the linearized equation of continuity, Eq. (E-10b). We first differentiate (E-13c, d) with respect to time:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \bar{\rho}\,\frac{d}{dy}\left(\frac{1}{\bar{\rho}}\right)\frac{\partial^2 V_z}{\partial t^2} - \frac{g}{\bar{\rho}}\,\frac{\partial^2 \rho'}{\partial z\partial t} = 0\ , \qquad \text{(E-14a)}$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}\right) - \bar{\rho}\,\frac{d}{dy}\left(\frac{1}{\bar{\rho}}\right)\frac{\partial^2 V_x}{\partial t^2} - \frac{g}{\bar{\rho}}\,\frac{\partial^2 \rho'}{\partial x\partial t} = 0\ . \qquad \text{(E-14b)}$$

From Eq. (E-10b)

$$\frac{\partial^2 \rho'}{\partial z\partial t} = -\,\frac{d\bar{\rho}}{dy}\,\frac{\partial V_y}{\partial z}$$

$$\frac{\partial^2 \rho'}{\partial x\partial t} = -\,\frac{d\bar{\rho}}{dy}\,\frac{\partial V_y}{\partial x}\ ,$$

which when substituted in Eq. E-14 gives the following pair of equations:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \frac{N^2}{g}\frac{\partial^2 V_z}{\partial t^2} - N^2 \frac{\partial V_y}{\partial z} = 0 , \tag{E-15a}$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}\right) - \frac{N^2}{g}\frac{\partial^2 V_x}{\partial t^2} - N^2 \frac{\partial V_y}{\partial x} = 0 , \tag{E-15b}$$

where N is the Brunt-Väisälä frequency given by

$$N^2 = -\frac{g}{\bar{\rho}}\left(\frac{d\bar{\rho}}{dy}\right) . \tag{E-16}$$

As the final step, differentiate Eq. (E-15a) and (E-15b) with respect to z, x, respectively. Adding the resulting equations we obtain

$$\frac{\partial^2}{\partial t^2}\left[\left(\frac{\partial}{\partial y}\nabla_T \cdot \underline{V} - \nabla_T^2 V_y\right) - \frac{N^2}{g}\nabla_T \cdot \underline{V}\right] - N^2 \nabla_T^2 V_y = 0 ,$$

where

$$\nabla_T = \underline{x}_o \frac{\partial}{\partial x} + \underline{z}_o \frac{\partial}{\partial z} .$$

Since by virtue of Eq. (E-1c) $\nabla_T \cdot \underline{V} = -\frac{\partial V_y}{\partial y}$, the preceding is equivalent to

$$\frac{\partial^2}{\partial t^2}\left[\nabla^2 V_y - \frac{N^2}{g}\frac{\partial V_y}{\partial y}\right] + N^2 \nabla_T^2 V_y = 0 , \tag{E-17}$$

which is the wave equation for the vertical velocity of small amplitude internal waves. The other two velocity components may

be obtained from V_y with the aid of Eqs. (E-11a) and (E-11b) together with the condition of incompressibility. One finds

$$\frac{\partial}{\partial t}\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right] = 0 \ , \tag{E-18}$$

$$\frac{\partial V_z}{\partial z} + \frac{\partial V_x}{\partial x} = - \frac{\partial V_y}{\partial y} \ , \tag{E-19}$$

the first of which is obtained from Eqs. (E-11a) and (E-11b) by differentiating with respect to x and z and eliminating $\frac{1}{\bar{\rho}} \frac{\partial^2 p'}{\partial x \partial z}$.

Equation (E-18) states that the vertical vorticity component is not an explicit function of time. This component may, therefore, be set equal to zero, since it can have no effect on the time-dependent internal wave motion. A nonzero vertical vorticity component can, however, be induced by viscous forces and by the effects of the earth's rotation. In the latter case the right side of Eq. (E-18) must be replaced by $2f \frac{\partial V_y}{\partial y}$, where f is the inertial frequency [14]. The wave equation (E-17) must then be modified by appending on the right the term

$$f^2 \left(\frac{N^2}{g} \frac{\partial V_y}{\partial y} - \frac{\partial^2 V_y}{\partial y^2}\right) .$$

The entire range of significant internal wave phenomena is encompassed within the (radian) frequency band $f < \omega < N_{max}$. Typical values of N_{max} are $.5 \times 10^{-2}$ rad/sec. The value of f varies from 1.4×10^{-4} rad/sec^{-1} at the poles to zero at the equator. We shall be interested only in frequencies substantially above the inertial frequency so that for our purposes $f = 0$.

The fundamental wave equation (E-17) differs from that given by Phillips [12] in that his result does not include the term $\frac{N^2}{g}\frac{\partial V_y}{\partial y}$. This term is retained by Roberts [14] and Krauss [15]. Equation (E-17) appears to have been first obtained in this form by Love [16]. Comparing our derivation with that of Phillips, one finds that Phillips sets $\bar{\rho} = \bar{\rho}_o$ = constant in the two momentum equations for the horizontal velocity components, i.e., our Eqs. (E-11a) and (E-11b). This eliminates the term $\frac{d}{dy}\left(\frac{1}{\bar{\rho}}\right)$ multiplying the pressure gradients in Eqs. (E-12c) and (E-12d), which term then does not appear in Eqs. (13) and (14). We then obtain the wave equation

$$\frac{\partial^2}{\partial t^2}\left[\nabla^2 V_y\right] + N^2\nabla_T^2 V_y = 0 \, , \tag{E-20}$$

which is Phillip's result. It is valid under the proviso that

$$\nabla^2 V_y >> \frac{N^2}{g}\frac{\partial V_y}{\partial y} \, . \tag{E-21}$$

The significance of this restriction is best examined in terms of the eigenvalue problem for the internal wave modes. We therefore first obtain a representation of the solution to Eq. (E-17) in terms of eigenfunctions in the y domain. The most direct approach is first to express V_y (x,z,y,t) as a bi-dimensional Fourier transform with respect to the transverse coordinates x,z (in the sequel collectively designated by the vector $\underline{\rho}$). Thus,

$$V_y(\underline{\rho},y,t) = \iint_{-\infty}^{\infty} e^{-i\underline{K}\cdot\underline{\rho}}\, \hat{V}_y(\underline{K},y,t)\, d^2\underline{K} \, . \tag{E-22}$$

Substituting this in Eq. (E-17) one finds that $\hat{V}_y\ (\underline{K},y,t)$ may be represented by

$$\hat{V}_y\ (\underline{K},y,t) = \exp \frac{1}{2g} \int_0^y N^2(\eta)\ d\eta$$

$$\cdot \sum_n \left[A_n^+(\underline{K})\ e^{i\Omega_n(K)t} + A_n^-(\underline{K})\ e^{-i\Omega_n(K)t} \right] \phi_n(y)\ , \qquad \text{(E-23)}$$

provided the $\phi_n(y)$ are chosen as solutions of the eigenvalue. equation

$$\frac{d^2}{dy^2}\ \phi_n(y) + \left[\frac{d}{dy}\left(\frac{N^2}{2g}\right) - \left(\frac{N^2}{2g}\right)^2 + K^2 \left(\frac{N^2}{\Omega_n^2} - 1\right) \right] \phi_n(y) = 0\ . \qquad \text{(E-24)}$$

The eigenvalues $\Omega_n(K)$ are angular frequencies that determine the dispersion relation for each component internal wave. The $A_n^+(\underline{K})$ and $A_n^-(\underline{K})$ are the two arbitrary constants associated with the second-order initial value problem in the time domain. If the boundary conditions at the endpoints $y = 0$ and $y = -D$ are of the form

$$\alpha \frac{\partial \phi}{\partial y} + \phi = 0\ , \qquad \text{(E-25)}$$

with α any real constant, then the boundary value problem in the y domain is hermitian, and the eigenfunctions $\phi_n(y)$ form a complete orthonormal set [17] (excepting, of course, some pathological $N(y)$ profiles, devoid of physical meaning). Reference [17] provides an extensive compendium of solved one-dimensional eigenvalue problems as well as techniques for determining eigenfunctions from the associated characteristic Green's function. For internal wave modes one usually assumes that the

vertical velocity at the surface and ocean bottom vanishes, so that $\alpha = 0$. Since in a deep ocean the Väisälä frequency decreases monotonically at large depths, a mathematically convenient boundary condition is lim $\phi_n(y) \to 0$ as $y \to -\infty$. An example is afforded by a Väisälä frequency profile that decreases exponentially with depth. Another possible boundary condition at $y = 0$ is the free surface condition for small vertical displacements of the ocean surface. This boundary condition is of the homogeneous form (E-24) wherein $\alpha \neq 0$. One finds that in this case Eq. (E-24) yields one solution that is independent of N, and which solution corresponds to small amplitude surface waves.

Although the set of eigenfunctions for the stated boundary conditions is complete, it will not necessarily be purely discrete. In case of a continuous spectrum, the sum in Eq. (E-23) must be replaced by an integral over a continuous parameter. Whether the spectrum is purely discrete or partly discrete and partly continuous depends on the combination of boundary conditions and the functional form of N(y). Analytical techniques for determining the spectral decomposition are presented in [17] and [18]. Generally, for profiles that are chosen to model internal wave phenomena in the ocean, purely discrete spectra are obtained.

We now return to the question posed earlier with regard to the quantitative significance of the differences between Eq. (E-17) and Eq. (E-20). Based on (E-20),

$$\hat{V}_y(\underline{K},y,t) = \sum \left[A_n^+(\underline{K})\, e^{i\Omega_n(K)t} + A_n^-(\underline{K})\, e^{i\Omega_n(K)t} \right] \phi_n(y) \qquad \text{(E-26)}$$

and the eigenvalue equation simplifies to

$$\frac{d^2}{dy^2}\phi_n(y) + K^2\left(\frac{N^2}{\Omega_n^2} - 1\right)\phi_n(y) = 0 \, . \qquad \text{(E-27)}$$

Thus, one consequence of retaining $\frac{N^2}{g} \frac{\partial V_y}{\partial y}$ in Eq. (E-17) is to introduce the multiplier $\exp \frac{1}{2g} \int_0^y N^2(\eta)\, d\eta$. Since $N_{max} \sim 5 \times 10^{-3}$ it is clear that for all practical purposes this factor is equal to unity. However, the modification introduced in the eigenvalue problem could be significant, since the "modified" profile in Eq. (E-24) contains a derivative of the Väisälä frequency. Clearly, when the profile varies rapidly with y, Eq. (E-24) instead of Eq. (E-27) should be employed. Actual ocean thermocline profiles are not expected to exhibit sufficiently abrupt spatial variations so as to give rise to significant differences between the eigenfunctions in Eq. (E-24) and Eq. (E-27). Care must be exercised, however, when profiles with abrupt changes are employed as mathematical models. A case in point is the constant multiple layer profile. Eigenfunction solutions in this case can, of course, also be obtained by a direct solution of the Laplace's equation in each layer and the application of boundary conditions at the interfaces, viz., without resorting to the formulation of the eigenvalue problem for the general spatially dependent profile. If, however, the latter formulation is used, then the correct equation is Eq. (E-24) and not Eq. (E-27).

Another instance illustrating the difference in the solutions of (E-24) and (E-27) arises when at the upper boundary the eigenfunctions are required to satisfy the linearized free surface boundary condition. One then finds that one of the solutions of (E-24) is a surface wave which, however, is not contained in the solutions of (E-27).

We shall only be concerned with Väisälä frequency profiles that are slowly varying and employ Eq. (E-26) and Eq. (E-27) in the subsequent theoretical discussion.

The two horizontal-velocity components may be found with the aid of Eq. (E-18) and Eq. (E-19). Employing the Fourier transform representation with respect to the transverse coordinates

$$V_{x,z}(\underline{\rho},y,t) = \iint_{-\infty}^{\infty} e^{-i\underline{K}\cdot\underline{\rho}}\, \hat{V}_{x,z}(\underline{K},y,t)\, d\underline{K}^2 ,$$

in conjunction with (E-22) in (E-18) and (E-19) yields

$$\hat{V}_x(\underline{K},y,t) = -\frac{iK_x}{K^2}\sum_n \dot{\phi}_n(y)\left[A_n^+(\underline{K})e^{i\Omega_n(K)t} + A_n^-(\underline{K})e^{-i\Omega_n(K)t}\right], \tag{E-28}$$

$$\hat{V}_z(\underline{K},y,t) = -\frac{iK_z}{K^2}\sum_n \dot{\phi}_n(y)\left[A_n^+(\underline{K})e^{i\Omega_n(K)t} + A_n^-(\underline{K})e^{-i\Omega_n(K)t}\right], \tag{E-29}$$

where $\dot{\phi}_n(y) \equiv \frac{d}{dy}\phi_n(y)$. Another quantity of interest is the vertical displacement of water particles defined by

$$\frac{\partial\eta}{\partial t} = V_y(\underline{\rho},y,t) . \tag{E-30}$$

With

$$\eta(\underline{\rho},y,t) = \iint_{-\infty}^{\infty} e^{-i\underline{K}\cdot\underline{\rho}}\, \hat{\eta}(\underline{K},y,t)\, d^2\underline{K} , \tag{E-31}$$

one finds from (E-26)

$$\hat{\eta}(\underline{K},y,t) = \sum_n \phi_n(y)\left[\frac{A_n^+(\underline{K})}{i\Omega_n} e^{i\Omega_n t} - \frac{A_n^-(\underline{K})}{i\Omega_n} e^{-i\Omega_n t}\right]. \tag{E-32}$$

B. EQUATIONS FOR LINEAR INTERNAL WAVES WITH VISCOSITY EFFECTS INCLUDED

We now examine the form assumed by the linear internal wave equation for the vertical velocity when viscosity is taken into account. Instead of (E-1a), one must start with the Navier-Stokes momentum equation

$$\rho \frac{\partial \underline{V}}{\partial t} + \rho \underline{V} \cdot \nabla \underline{V} + \nabla p + \rho g \underline{y}_o = \mu \nabla^2 \underline{V} . \qquad \text{(E-33)}$$

If we again assume that the mean fluid velocity is zero (viz., Eq. E-4), the small perturbation argument employed in the inviscid case remains unaltered. Instead of (E-10a) the linearized momentum equation reads

$$\frac{\partial \underline{V}}{\partial t} + \frac{\nabla p'}{\bar{\rho}} + \frac{\rho'}{\bar{\rho}} g \underline{y}_o = \left(\frac{\mu}{\bar{\rho}}\right) \nabla^2 \underline{V} , \qquad \text{(E-34)}$$

with E-10b and E-1c remaining, of course, unaltered. As a notational convenience, let

$$\underline{F} = \frac{\mu}{\bar{\rho}} \nabla^2 \underline{V} = \nu \nabla^2 \underline{V} . \qquad \text{(E-35)}$$

We will take ν as a constant (for water at 68°, $\nu \approx 10^{-6}$ m^2/sec). After differentiating E-34 as in E-12 and making the substitutions for p' yields the generalization of (E-13):

$$\frac{\partial}{\partial t}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \bar{\rho} \frac{d}{dy}\left(\frac{1}{\bar{\rho}}\right)\frac{\partial V_z}{\partial t} - \frac{g}{\bar{\rho}}\frac{\partial \rho'}{\partial z} = - \bar{\rho} \frac{d}{dy}\left(\frac{1}{\bar{\rho}}\right) F_z + \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) , \qquad \text{(E-36)}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}\right) - \bar{\rho} \frac{d}{dy}\left(\frac{1}{\bar{\rho}}\right)\frac{\partial V_x}{\partial t} - \frac{g}{\bar{\rho}}\frac{\partial \rho'}{\partial x} = - \bar{\rho} \frac{d}{dy}\left(\frac{1}{\bar{\rho}}\right) F_x + \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right) . \qquad \text{(E-37)}$$

By following the same steps as in (E-14)-(E-17) we obtain

$$\frac{\partial^2}{\partial t^2}\left[\nabla^2 V_y - \frac{N^2}{g}\frac{\partial V_y}{\partial y}\right] + N^2\nabla_T^2 V_y = -\frac{N^2}{g}\frac{\partial}{\partial t}\nabla_T \cdot \underline{F} + \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right] + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}F_x - \frac{\partial}{\partial x}F_y\right]\right\}.$$

(E-38)

The left side is, of course, precisely the same as (E-17). We shall now cast the left side into a simpler form. Employing (E-35) and assuming ν a constant, one can readily show that the last term in (E-38) equals $-\nu\nabla^2\nabla^2 \frac{\partial V_y}{\partial t}$; also $\nabla_T \cdot \underline{F} = -\nu\nabla^2 \frac{\partial V_y}{\partial y}$. The final result, therefore, becomes

$$\frac{\partial^2}{\partial t^2}\left[\nabla^2 V_y - \frac{N^2}{g}\frac{\partial V_y}{\partial y}\right] + N^2\nabla_T^2 V_y - \nu\frac{N^2}{g}\frac{\partial}{\partial t}\nabla^2\frac{\partial V_y}{\partial y} + \nu\frac{\partial}{\partial t}\nabla^2\nabla^2 V_y = 0 . \qquad \text{(E-39)}$$

The other two velocity components follow from the condition of incompressibility and the two horizontal momentum equations:

$$\frac{\partial V_z}{\partial z} + \frac{\partial V_x}{\partial x} = -\frac{\partial V_y}{\partial y} , \qquad \text{(E-40a)}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) = -\nu\nabla^2\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) . \qquad \text{(E-40b)}$$

Note that $\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}$ is the vertical component of vorticity. In the inviscid case, this component was found to be identically zero. This is no longer true when viscous effects are included.*
If we now assume a solution of the form

$$V_y(\underline{\rho}, y) = \sim\iint_{-\infty}^{\infty} \phi(y)A(\underline{K})e^{-i\underline{K}\cdot\underline{\rho} + i\Omega(K)t}d^2\underline{K} ,$$

*Equation 40b will be recognized as the linearized version of the transport equation for the vertical vorticity component.

then the one-dimensional equation for ϕ reads

$$\left(\frac{\nu}{i\Omega}\right)\frac{d^4\phi}{dy^4}-\left(\frac{\nu N^2}{i\Omega g}\right)\frac{d^3\phi}{dy^3}+\left(1-\frac{2\nu K^2}{i\Omega}\right)\frac{d^2\phi}{dy^2}-\frac{N^2}{g}\left(1-\frac{\nu K^2}{i\Omega}\right)\frac{d\phi}{dy}+K^2\left(\frac{N^2}{\Omega^2}-1+\frac{\nu K^2}{i\Omega}\right)\phi=0\,. \tag{E-41}$$

This fourth-order equation is of the Orr-Sommerfeld type [19] The theory underlying its solution is substantially more complicated than the theory underlying the second order Storm-Lionville equation obtained in the inviscid case. Although the Orr-Sommerfeld equation is extensively discussed in the mathematical literature, its application to the study of internal waves in the ocean appears not to have received much attention. It is important to note that even though the viscosity coefficient ν is exceedingly small ($\nu \approx 10^{-6}$ m²/sec), the solution of Eq. (E-41) can in general not be obtained by simply setting $\nu = 0$ since thereby the order of the differential equation is reduced from the fourth to the second. To obtain solutions for small ν, one must resort to the techniques of singular perturbations [19]. We shall not attempt to carry out the rather intricate mathematical development at this time. However, on the basis of available theory [19] one can state that the effects of viscosity will be strongest in the regions of high wave numbers, particularly in the vicinity of turning points of the differential equation. In addition, the whole question of mode completeness which is so straightforward in the inviscid case, presents several delicate and as yet unresolved mathematical problems.

C. EXCITATION OF LINEAR INTERNAL WAVES

In (a) we derived the homogeneous equation governing the propagation of linear internal waves. The energy sources of such internal waves have not been included. A technique that is sometimes employed is to assume a source function on the right of Eq. (E-17). For several reasons this is not a physically

satisfactory procedure. First, the source must be postulated so that the underlying assumptions that had been made in linearizing the wave equation are not violated. Second, since in the derivation of the wave equation the mean velocity field was taken as zero, only sources which yield zero mean velocity fields are permissible. It is difficult to determine a priori what constraints must be imposed on the source term to satisfy these two conditions. To account for internal wave sources and at the same time retain a consistent framework of a linear theory, one can consider the internal wave velocities as small perturbations about a mean velocity field. We briefly explore this alternative.

Returning to the fundamental equations (E-1), we now reformulate the problem for the case of nonzero mean velocity fields. For simplicity we exclude viscosity effects. We now let

$$\underline{V} = \overline{\underline{V}} + \underline{V} ,$$

where $\underline{V}$ is the perturbation about the mean $\overline{\underline{V}}$. The other symbols on pages 225-226 remain unaltered. Equation (E-2) now reads

$$\overline{\rho} \frac{\partial \overline{\underline{V}}}{\partial t} + \overline{\rho} \frac{\partial \underline{V}}{\partial t} + \overline{\rho}(\overline{\underline{V}} + \underline{V}) \cdot \nabla(\overline{\underline{V}} + \underline{V}) + \nabla(\overline{p} + p')$$

$$+ (\rho' + \overline{\rho}) g\underline{y}_o = 0 . \qquad \text{(E-42)}$$

In the spirit of the usual perturbation argument we equate terms having the same orders of magnitude. The momentum equation in $\overline{\underline{V}}$ is

$$\overline{\rho} \frac{\partial \overline{\underline{V}}}{\partial t} + \overline{\rho} (\overline{\underline{V}} \cdot \nabla\overline{\underline{V}}) + \overline{\rho} g\underline{y}_o \qquad \text{(E-43)}$$

$$+ \nabla(\overline{p}) = 0 .$$

Equating linear terms in $\underline{V}$, ρ' and p' in Eq. (E-42) yields

$$\bar{\rho}\,\frac{\partial \underline{V}}{\partial t} + \bar{\rho}[(\underline{\bar{V}} \cdot \nabla\underline{V}) + \underline{V} \cdot \nabla\underline{\bar{V}}] + \nabla p'$$

$$+ \rho' g \underline{y}_o = 0 \ . \qquad \text{(E-44)}$$

Comparing this with Eq. (E-10a), we note that Eq. (E-44) contains two terms coupling to the mean flow. These terms will now act as sources for linear internal waves. Next, we turn to the continuity equation, which now becomes

$$(\underline{V} + \underline{\bar{V}}) \cdot \nabla(\bar{\rho} + \rho') + \frac{\partial}{\partial t}(\bar{\rho} + \rho') = 0 \ .$$

Equating terms of the first two orders of smallness gives

$$\underline{\bar{V}} \cdot \nabla\bar{\rho} + \frac{\partial\bar{\rho}}{\partial t} = 0 \ , \qquad \text{(E-45)}$$

$$\underline{V} \cdot \nabla\bar{\rho} + \underline{\bar{V}} \cdot \nabla\rho' + \frac{\partial\rho'}{\partial t} = 0 \ . \qquad \text{(E-46)}$$

If we assume stratification of the density only in the y-direction, these become

$$\bar{V}_y \frac{\partial\bar{\rho}}{\partial y} + \frac{\partial\bar{\rho}}{\partial t} = 0 \ , \qquad \text{(E-47)}$$

$$V_y \frac{\partial\bar{\rho}}{\partial y} + \underline{\bar{V}} \cdot \nabla\rho' + \frac{\partial\rho'}{\partial t} = 0 \ . \qquad \text{(E-48)}$$

Note that consistency of the perturbation procedure requires $\frac{\partial \bar{\rho}}{\partial t} \neq 0$ unless $\frac{\partial \bar{\rho}}{\partial y} = 0$ (no stratification) or $\bar{V}_y = 0$ (zero mean flow in the vertical direction). Thus, Eq. (E-10b) must now be replaced by Eq. (E-47) and Eq. (E-48). The incompressibility condition now applies to $\underline{\bar{V}}$ and $\underline{V}$ separately, i.e.,

$$\nabla \cdot \underline{\bar{V}} = 0. \tag{E-49}$$

$$\underline{\nabla} \cdot \underline{V} = 0 . \tag{E-50}$$

Thus, the *linear* equations for $\underline{V}$ in a stratified ocean are

$$\bar{\rho} \frac{\partial V}{\partial t} + \nabla p' + \rho' g \underline{y}_o = - \bar{\rho}(\underline{\bar{V}} \cdot \nabla \underline{V} + \underline{V} \cdot \nabla \underline{\bar{V}}), \tag{E-51}$$

$$V_y \frac{\partial \bar{\rho}}{\partial y} + \frac{\partial \rho'}{\partial t} = - \underline{\bar{V}} \cdot \nabla \rho' , \tag{E-52}$$

$$\nabla \cdot \underline{\bar{V}} = 0 . \tag{E-53}$$

If the mean velocity $\underline{\bar{V}}$ were zero, the right side of Eq. (E-51) and Eq. (E-52) would also be zero, leading to the standard equations for linear internal waves. With $\underline{\bar{V}} \neq 0$ the situation is substantially more complicated, even though the equations are perfectly linear. In general, these equations cannot be reduced to a set of wave equations for V_x, V_y, V_z, mainly because the source terms arising from the mean flow appear as coefficients in the partial differential equation for $\underline{V}$. Moreover, in order to transfer energy to the linear field $\underline{V}$, these coefficients

must depend explicitly on time. Only in certain special cases (e.g., where $\underline{V}$ does not depend on time and has only horizontal components) does one find that V_y satisfies a characteristic equation for internal wave eigenmodes. In this special case the horizontal velocities are referred to as mean shear. The equivalent profile is then found to depend on the horizontal components of $\bar{\underline{V}}$, thus substantially increasing the difficulty in obtaining analytical solutions for the eigenfunctions [14], [20]. Nevertheless, if one wishes to include source terms that are compatible with the linearized theory, one must start with Eqs. (E-51 - E-52). Note that the problem of solving for $\bar{\underline{V}}$ is a separate affair. The equations governing $\bar{\underline{V}}$ are in general nonlinear and might have to be solved numerically.

II. STATISTICAL DESCRIPTION OF LINEAR INTERNAL WAVES

It is generally recognized that the complexity of ocean current dynamics compels a statistical description of internal wave phenomena. At the same time, a statistical model with a useful predictive capability must be based on a hydrodynamic description that holds good for a typical realization of an internal wave stochastic process. Precisely what tradeoffs between physical realism and mathematical simplicity are permissible can in the final analysis be decided only by reference to experimental data. Since on the one hand a full nonlinear description is far too complex, while on the other hand linear modeling has apparently yielded some agreement with experimental data [14], we shall restrict ourselves to the purely linear case. Even in the purely linear model there are several possible levels of complexity (presumably related to the degree of physical realism). Thus, the most complex linear model would incorporate mean shear and viscosity, while the simplest would neglect both of these effects, and include only the dependence on the Väisälä frequency profile plus some reasonable assumption on the modal excitation coefficients. Here we shall confine ourselves only to the simplest case, and, moreover, formulate the problem at the outset by treating the internal wave field as a temporally stationary and spatially homogeneous stochastic process. This almost naive approach certainly makes up in simplicity for what it undoubtedly loses in physical realism. These postulates are, of course, not without precedent. They appear to be implicit in past statistical treatments of ocean internal phenomena, such as, e.g., in the work of Garrett and Munk [8].

The essential difference between their theory and the one presented herein is that the statistical postulates are incorporated into the stochastic model *ab initio*, and the consequences of any additional assumptions are treated within the framework of a systematic deductive scheme. The approach parallels closely that presented by Milder [9].

A. CORRELATION FUNCTIONS AND SPECTRA

We shall suppose that the fluid velocity vector components V_x, V_y, V_z are stationary random processes with spatially homogeneous second moments in the transverse (x,z) plane. We define cross-correlation functions

$$R_{\nu\mu}(\underline{\rho},y,\tau) = \langle V_\nu(\underline{\rho}'+\underline{\rho},y,t+\tau) \; V_\mu^* \; (\underline{\rho}',y,t)\rangle , \qquad \text{(E-54)}$$

where $\nu = x,y,z$ $\mu = x,y,z$, and the corresponding temporal cross-spectra by

$$\Phi_{\nu\mu}(\underline{\rho},y,\omega) = \int_{-\infty}^{\infty} R_{\nu\mu}(\underline{\rho},y,\tau) \; e^{-i\omega\tau} \, d\tau \; . \qquad \text{(E-55)}$$

Substituting (E-26) in (E-54) yields

$$R_{yy}(\underline{\rho},y,\tau) =$$

$$\iint_{-\infty}^{\infty} d^2\underline{K} \iint_{-\infty}^{\infty} d^2\underline{K}' \; e^{-i\underline{K}\cdot(\underline{\rho}'+\underline{\rho}) \; + \; i\underline{K}'\cdot\underline{\rho}'} \sum_{nm} \phi_n \phi_m' \;\; a_{nm}(\underline{K},\underline{K}';t,\tau), \qquad \text{(E-56)}$$

where

$$a_{nm}(\underline{K},K',t,\tau) =$$

$$\langle [A_n^+(\underline{K}) \; e^{i\Omega_n(t+\tau)} + A_n^-(\underline{K})\, e^{-i\Omega_n(t+\tau)}] \, [A_m^{+*}(\underline{K}') \; e^{-i\Omega_m t} + A_m^{-*}(\underline{K}') \; e^{i\Omega_m t}]\rangle .$$

Temporal stationarity and spatial homogeneity requires (Cf. Appendix A, Eq. (A-25)) that

$$\langle A_n^+(\underline{K})\ \overset{*}{A}{}_m^+(\underline{K}')\rangle = \frac{1}{2}\ \delta_{nm}\ \delta(\underline{K}-\underline{K}')\ \psi_n^+(\underline{K}) \quad ,$$

$$\langle A_n^-(\underline{K})\ \overset{*}{A}{}_m^-(\underline{K}')\rangle = \frac{1}{2}\ \delta_{nm}\ \delta(\underline{K}-\underline{K}')\ \psi_n^-(\underline{K}) \quad ,$$

$$\langle A_n^-(K)\ \overset{*}{A}{}_m^+(\underline{K}')\rangle = \langle A_n^+(\underline{K})\ \overset{*}{A}{}_m^-(\underline{K}')\rangle = 0 \quad , \qquad \text{(E-56')}$$

i.e., the excitation amplitudes of the normal modes are uncorrelated for different mode indexes and different wavenumbers. The functions $\psi_n^+(\underline{K})$ and $\psi_n^-(\underline{K})$ are spatial spectral amplitudes. By an argument identical to that employed in Appendix A for surface waves one can show that they are both real and obey the point reflection symmetry in wavenumber space:

$$\psi_n^+(\underline{K}) = \psi_n^-(-\underline{K}) \ .$$

We shall henceforth write $\psi_n(\underline{K})$ for $\psi_n^+(\underline{K})$. Employing the above relations in (E-56) yields

$$R_{yy}(\underline{\rho},y,\tau) = \frac{1}{2}\iint_{-\infty}^{\infty} d^2\underline{K}\ e^{-i\underline{K}\cdot\underline{\rho}} \sum_n \phi_n^2(y)[\psi_n(\underline{K})\, e^{i\Omega_n\tau} + \psi_n(-\underline{K})\ e^{-i\Omega_n\tau}] \ . \qquad \text{(E-57)}$$

Taking the Fourier transform and changing to polar coordinates yields for $\omega>0$

$$\Phi_{yy}(\underline{\rho},y,\omega) = \pi\sum_n \int_0^{2\pi} d\varpi\ e^{-iK_n(\omega)\rho\cos(\varpi-\theta)}\ \frac{K_n(\omega)\ \phi_n^2(y)}{\left.\frac{d}{dK}\,\Omega_n(K)\right|_{K=K_n(\omega)}}\psi_n[K_n(\omega),\varpi]. \qquad \text{(E-58)}$$

Note that

$$\frac{d}{dK}\,\Omega_n(K)\bigg|_{K=K_n(\omega)} \equiv v_{gn}(\omega)$$

is the group speed of the n^{th} mode evaluated at frequency ω. Similarly, one finds

$\Phi_{xx}(\underline{\rho},y,\omega) =$

$$\pi \sum_n \int_0^{2\pi} dw\; e^{-iK_n(\omega)\rho\cos(w-\theta)} \frac{\dot{\phi}_n^2(y)\cos^2 w}{K_n(\omega)\; v_{gn}(\omega)}\; \psi_n\,[K_n(\omega),w]\;, \quad \text{(E-59)}$$

$\Phi_{zz}(\underline{\rho},y,\omega) =$

$$\pi \sum_n \int_0^{2\pi} dw\; e^{-iK_n(\omega)\rho\cos(w-\theta)} \frac{\dot{\phi}_n^2(y)\sin^2 w}{K_n(\omega)\; v_{gn}(\omega)}\; \psi_n\,[K_n(\omega),w]\;. \quad \text{(E-60)}$$

The correlation function of the displacement η is

$$R_{\eta\eta}(\underline{\rho},y,\tau) = \langle \eta(\underline{\rho}'+\underline{\rho},y,t+\tau)\; \eta^*(\underline{\rho}',y,t)\rangle \;,$$

with the corresponding cross-spectrum

$$\Phi_{\eta\eta}(\underline{\rho},y,\omega) = \pi \sum_n \int_0^{2\pi} dw\; e^{-iK_n(\omega)\rho\cos(w-\theta)} \frac{\phi_n^2(y)\; K_n(\omega)}{\omega^2\; v_{gn}(\omega)}\; \psi_n[K_n(\omega),\, w].$$

(E-61)

B. ENERGY RELATIONS

The total kinetic energy per unit horizontal surface area is, by definition,

$$T = \frac{1}{2}\,\rho_o \int_{-D}^{0} [V_x^2(\underline{r},t) + V_y^2(\underline{r},t) + V_z^2(\underline{r},t)]\; dy \;, \quad \text{(E-62)}$$

where we have assumed the density $\rho \approx \bar{\rho}$ constant. Similarly, the potential energy per unit horizontal surface area is

$$W = \frac{1}{2}\rho_o \int_{-D}^{0} N^2(y)\ \eta^2(\underline{r},t)\ dy\ . \qquad \text{(E-63)}$$

We now compute the averages $\langle T \rangle$ and $\langle W \rangle$.

$$\langle T \rangle = \frac{1}{2}\rho_o \int_{-D}^{0} [\langle V_x^2 + V_y^2 + V_z^2 \rangle]\ dy$$

$$= \frac{1}{2}\rho_o \int_{-D}^{0} dy \cdot \int_{-\infty}^{\infty} [\Phi_{xx}(0,y,\omega) + \Phi_{yy}(0,y,\omega) + \Phi_{zz}(0,y,\omega)]\ d\omega$$

$$= \int_{-\infty}^{\infty} d\omega\ \frac{\rho_o}{2} \int_{-D}^{0} [\Phi_{xx}(0,y,\omega) + \Phi_{yy}(0,y,\omega) + \Phi_{zz}(0,y,\omega)]\ dy$$

$$= \int_{-\infty}^{\infty} E_T(\omega)\ d\omega\ . \qquad \text{(E-64)}$$

We would like to obtain a representation of the temporal kinetic energy spectrum $E_T(\omega)$ in terms of the spectral mode excitation functions $\psi_n(K_n,w)$. For this purpose we shall need certain orthogonality relations for the mode amplitudes. First, with the boundary conditions $\phi_n(0) = \phi_n(-D) = 0$, (E-27) yields a complete orthogonal set. We normalize the ϕ_n as follows:

$$\int_{-\infty}^{0} \phi_n \phi_m N^2 dy \equiv (\phi_n,\ \phi_m N^2) = \delta_{nm}\ . \qquad \text{(E-65)}$$

Second, multiplying (E-27) by ϕ_m and integrating by parts we obtain

$$\phi_m \dot{\phi}_n \Big|_{-D}^{0} - (\dot{\phi}_m, \dot{\phi}_n) + \frac{K^2}{\Omega_n^2}\ \delta_{nm} = K^2(\phi_n, \phi_m)\ . \qquad \text{(E-66)}$$

Since ϕ_m vanishes at the indicated limits, the preceding may be written

$$K^2(\phi_n,\phi_m) + (\dot{\phi}_n,\dot{\phi}_m) = \frac{K^2}{\Omega_n^2}\ \delta_{mn} \ . \tag{E-67}$$

We now evaluate the integral over y in (E-64):

$$\int_{-D}^{0} \Phi_{yy}(0,y,\omega)\ dy + \int_{-D}^{0} \Phi_{zz}(0,y,\omega)\ dy + \int_{-D}^{0} \Phi_{xx}(0,y,\omega)\ dy$$

$$= \pi \sum_n \int_0^{2\pi} dw\ \frac{\psi_n[K_n(\omega),w]}{v_{gn}(\omega)}\ [K_n(\omega)\ (\phi_n,\phi_n) + \frac{1}{K_n(\omega)}\ (\dot{\phi}_n,\dot{\phi}_n)] \ .$$

Upon employing (E-67) we obtain

$$E_T(\omega) = \frac{\pi\rho_o}{2} \sum_n \frac{K_n(\omega)}{\omega^2 v_{gn}(\omega)} \int_0^{2\pi} dw\ \psi_n[K_n(\omega),w] \ . \tag{E-68}$$

From (E-63), the mean potential energy is

$$\langle W\rangle = \frac{1}{2}\ \rho_o \int_{-D}^{0} N^2(y)\ \langle\eta^2(r,t)\rangle\ dy$$

$$= \frac{1}{2}\ \rho_o \int_{-D}^{0} N^2(y)\ \ dy \int_{-\infty}^{\infty} \Phi_{\eta\eta}(0,y,\omega)\ d\omega$$

$$= \int_{-\infty}^{\infty} d\omega\ \frac{1}{2}\ \rho_o \int_{-D}^{0} N^2(y)\ \Phi_{\eta\eta}(0,y,\omega)\ dy$$

$$= \int_{-\infty}^{\infty} d\omega\ E_w(\omega) \ ,$$

where

$$E_w(\omega) = \frac{1}{2} \rho_o \int_{-D}^{0} N^2(y)\, \Phi(0,y,\omega) \quad dy \quad . \tag{E-69}$$

Upon substituting (E-61) in (E-69) and employing the adopted normalization $(\phi_n,\phi_n N^2) = 1$ we obtain

$$E_w(\omega) = E_T(\omega) \quad .$$

This shows that if an ensemble of linear internal waves forms a stationary and homogeneous (in the transverse plane) stochastic process, then at a given frequency the mean kinetic and potential energies are equal. The total energy spectrum is then

$$\begin{aligned} E(\omega) &= 2E_T(\omega) = 2E_w(\omega) \\ &= \pi\rho_o \sum_n \frac{K_n(\omega)}{\omega^2 v_{gn}(\omega)} \int_0^{2\pi} dw\, \psi_n[K_n(\omega),w] \quad . \end{aligned} \tag{E-70}$$

The total energy integrated over all frequencies ω is:

$$E \equiv 2\int_0^{\infty} E(\omega)\, d\omega = \int_0^{2\pi} dw 2\pi\, \rho_o \sum_n \int_0^{\infty} \frac{K_n(\omega)\, d\omega\, \psi_n[K_n(\omega),w]}{\omega^2 v_{gn}(\omega)} \quad .$$

With $\Omega_n(K) = \omega$ in the nth integral, $d\omega = v_{gn}(\omega)\, dK_n$

$$\int_0^{\infty} \frac{K_n(\omega)\, d\omega\, \psi_n[K_n(\omega),w]}{\omega^2 v_{gn}(\omega)} = \int_0^{\infty} \frac{K_n dK_n\, \psi_n(K_n,w)}{\Omega_n^2(K_n)} \equiv \int_0^{\infty} KdK\, \frac{\psi_n(K,w)}{\Omega_n^2(K)} \quad .$$

Hence, E may also be written

$$E = 2\pi\rho_o \int_0^{\infty} KdK \int_0^{2\pi} dw \sum_n \left[\frac{\psi_n(K,w)}{\Omega_n^2(K)}\right] . \tag{E-71}$$

The quantity

$$\varepsilon(\underline{K}) = 2\pi\rho_o \sum_n \frac{\psi_n(K,w)}{\Omega_n^2(K)} \qquad \text{(E-72)}$$

is clearly the wavenumber energy spectrum.

C. INTERNAL WAVE SPECTRA UNDER MILDER'S ENERGY PARTITIONING HYPOTHESIS

The expressions for the velocity and displacement spectra, Eqs. (E-58) - (E-61), depend on the eigenfunctions, the associated dispersion relations, and the spectral excitation functions $\psi_n(\underline{K})$. While the Väisälä frequency profile (and hence individual eigenfunctions) as well as the total spectrum are (more or less) subject to experimental verification, a direct measurement of the relative distribution of energy in the mode wavenumber space, governed by $\psi_n(\underline{K})$, is much more difficult. On the theoretical level, the physical basis underlying the form of $\psi_n(\underline{K})$ must be sought in the mechanisms underlying the excitation of internal waves. Since the excitation mechanisms are at present not well understood, nor is it for that matter at all evident that the simplified linear theory is adequate for their description, some "bold" hypothesis is needed, the adequacy of which could subsequently be established by correlating its consequences with experiment. One such hypothesis is due to Milder [9], which asserts that in the equilibrium state deep ocean internal wave energy is distributed among the modes in proportion to the square of their individual phase velocities. For a physical rationale underlying this assumption the reader should consult Milder's paper.

From (E-71), Milder's assumption is expressed as

$$2\pi K \frac{\psi_n(K,w)}{\Omega_n^2(K)} = 2I(\underline{K}) \left[\frac{\Omega_n(K)}{K}\right]^2 , \qquad \text{(E-73)}$$

where $I(\underline{K})$ is an excitation function which, for the moment, we may leave unspecified. Note that $I(\underline{K})$ may in general exhibit directional properties in wavenumber space. The crucial assumption is not the precise functional form of $I(\underline{K})$ but its independence of the internal wave mode number n. One of the consequences of this assumption is that the spatial wavenumber spectra of energy as well as velocity components can be expressed explicitly in terms of the Väisälä frequency profile.

We first consider the wavenumber energy spectrum, Eq. (E-72). Employing (E-73) one has

$$\varepsilon(\underline{K}) = 2\rho_o I(\underline{K}) \frac{1}{K^3} \sum_{n=1}^{\infty} \Omega_n^2(K) \quad . \tag{E-74}$$

This sum can be expressed as an integral involving the Väisälä frequency. With the aid of identities derived in Appendix F, viz., Eq. (F-5) and (F-7), and the orthogonality condition (E-65), one obtains

$$\sum_{n=1}^{\infty} \Omega_n^2(K) = \frac{K}{2} \int_{-D}^{0} N^2(y)\, f_D(y)\, dy \ , \tag{E-75}$$

where

$$f_D(y) = \frac{\cosh KD - \cosh K(2y+D)}{\sinh KD} \quad . \tag{E-76}$$

If the ocean is assumed infinitely deep and the bottom boundary condition is taken as $\lim \phi_n(y) \to 0;\ y \to -\infty$, then the lower limit of integration in (E-75) may be taken as $-\infty$, and $f_D(y)$ replaced by*

$$f_\infty(y) = 1 - e^{2Ky} \quad . \tag{E-77}$$

*Note that f_∞ is not equal to $\lim f_D$ as $D \to -\infty$, since the limiting forms of the boundary conditions in the two cases are different (one is a limit circle, the other a limit point condition) [c.f., Eqs. (F-7) and (F-8)].

The wavenumber energy spectrum can now be written

$$\varepsilon(\underline{K}) = \rho_o \frac{I(\underline{K})}{K^2} \int_{-D}^{0} N^2(y) f_D(y)\, dy \; . \tag{E-78}$$

From (E-57), the spatial spectrum of the vertical velocity at $\tau = 0$ is

$$S_{yy}(\underline{K},y,0) = \frac{1}{2} \sum_n \phi_n^2(y) \, [\psi_n(\underline{K}) + \psi_n(-\underline{K})] \quad . \tag{E-79}$$

Similarly, for the vertical displacement one obtains

$$S_{\eta\eta}(\underline{K},y,0) = \frac{1}{2} \sum_n \frac{\phi_n^2(y)}{\Omega_n^2(K)} [\psi_n(\underline{K}) + \psi_n(-\underline{K})] \quad . \tag{E-80}$$

Again employing the energy partitioning hypothesis (E-73), one finds with the aid of (F-5)

$$S_{\eta\eta}(\underline{K},y,0) = \frac{1}{2\pi} \left[\frac{I(\underline{K}) + I(-\underline{K})}{2K^2}\right] f_D(y) \quad . \tag{E-81}$$

This formula is quite remarkable since it states that the spatial spectrum of the particle displacement is completely independent of the Väisälä frequency profile: the dependence of $S_{\eta\eta}$ on y is governed entirely by (E-76), or, in the deep ocean approximation, by (E-77). Moreover, the simple relationship between the spatial spectrum $S_{\eta\eta}$ and the excitation function $I(\underline{K})$ provides a means for its experimental determination.

The spatial spectrum of the vertical velocity in (E-79) can also be put in a form in which the direct dependence on the eigenfunctions is suppressed. Inserting (E-73) in (E-79) gives

$$S_{yy}(\underline{K},y,o) = \frac{1}{2\pi} \left[\frac{I(\underline{K}) + I(-\underline{K})}{K^3}\right] \sum_{n=1}^{\infty} \phi_n^2(y)\Omega_n^4(K) \quad . \tag{E-82}$$

Again from Appendix F, Eq. (F-6), one has

$$\sum_{n=1}^{\infty} \phi_n^2(y)\Omega_n^*(K) = K^* \int_{-D}^{0} N^2(y'') \, g^2(y'',y) \, dy'' , \qquad \text{(E-83)}$$

so that

$$S_{\dot{y}\dot{y}}(\underline{K},y,o) = \frac{1}{2\pi} [I(\underline{K}) + I(-\underline{K})] \, K \int_{-D}^{0} N^2(y'') \, g^2(y'',y) \, dy'' . \qquad \text{(E-84)}$$

Thus, the spatial spectrum of the vertical velocity is proportional to an integral involving the Väisälä frequency. Similar expressions can be obtained for the two horizontal velocity components. It is also interesting to observe that the ratio of the (spatial) vertical velocity spectrum to the spatial spectrum of the displacement is independent of the excitation function, viz.,

$$\frac{S_{\dot{y}\dot{y}}(\underline{K},y,o)}{S_{\eta\eta}(\underline{K},y,o)} = \frac{2K^3 \int_{-D}^{0} N^2(y'') \, g^2(y'',y) \, dy''}{f_D(y)} , \qquad \text{(E-85)}$$

which ratio is readily computed for a particular Väisälä frequency profile.

The suppression of the explicit dependence of the spatial spectra on the eigenfunctions and dispersion relations is a direct consequence of the assumption that the relative energy distribution among modes is in proportion to the squares of their phase velocities, but is completely independent of the nature of the excitation function I($\underline{K}$). On the other hand, the temporal cross-spectra, Eqs. (E-70), and (E-58) - (E-60), do not attain a similar simplification but instead depend explicitly on the eigenfunctions and the associated dispersion

relations. Employing (E-73) one finds the following relations for the temporal spectra of energy, displacement, and velocity:

$$E(\omega) = \rho_o \omega^2 \sum_n \frac{1}{[K_n(\omega)]^2 \, v_{gn}(\omega)} \int_0^{2\pi} dw I[K_n(\omega),w] \; , \tag{E-86}$$

$$\Phi_{\eta\eta}(0,y,\omega) = \omega^2 \sum_n \frac{\phi_n^2(y)}{K_n^2(\omega) v_{gn}(\omega)} \int_0^{2\pi} I[K_n(\omega),w] \, dw \; , \tag{E-87}$$

$$\Phi_{yy}(0,y,\omega) = \omega^4 \sum_n \frac{\phi_n^2(y)}{K_n^2(\omega) v_{gn}(\omega)} \int_0^{2\pi} I[K_n(\omega),w] \, dw$$

$$= \omega^2 \Phi_{\eta\eta}(0,y,\omega) \; , \tag{E-88}$$

$$\Phi_{xx}(0,y,\omega) = \pi\omega^4 \sum_n \frac{\dot{\phi}_n^2(y)}{[K_n(\omega)]^4 v_{gn}(\omega)} \int_0^{2\pi} I[K_n(\omega),w] \, dw$$

$$= \Phi_{zz}(0,y,\omega) \; . \tag{E-89}$$

It has been suggested that equilibrium ocean internal wave spectra are isotropic in wave number space [8][9]. An isotropic excitation function for which there appears some experimental justification is of the form

$$I(\underline{K}) = CK^{-p} \; , \tag{E-90}$$

where $1 \leq p \leq 2$, and C a constant. Consider first the rather artificial case of a constant Väisälä frequency profile. Then,

$$K_n(\omega) = \frac{n\pi}{D} \omega \frac{1}{\sqrt{(N^2-\omega^2)}} \; , \tag{E-91}$$

$$K_n v_{gn} = \frac{(N^2-\omega^2)}{N^2} \quad , \tag{E-92}$$

$$\phi_n(y) = \frac{1}{N}\sqrt{\frac{2}{D}} \sin\frac{n\pi y}{D} \quad . \tag{E-93}$$

The temporal frequency energy spectrum becomes

$$E(\omega) = 2\pi\rho_o\, C \left(\frac{D}{\pi}\right)^{p+1} N^2 \frac{[N^2-\omega^2]^{\frac{p-1}{2}}}{\omega^p} \left(\sum_{n=1}^{\infty} n^{-(p+1)}\right) . \tag{E-94}$$

The constant C can be determined from the normalization constraint on the total energy:

$$E = 2 \int_{\omega_i}^{N} E(\omega)\, d\omega \quad , \tag{E-95}$$

where ω_i is the low frequency cutoff (e.g., the inertial frequency). (The factor of 2 enters because (E-94) gives only the positive frequency part of the double-sided spectrum.) The constant C is then

$$C = \frac{E}{N^2 \left(\frac{D}{\pi}\right)^{(p+1)} 4\pi\rho_o\, \zeta(p+1)\, F_p\left(\frac{\omega_i}{N_c}\right)} \quad , \tag{E-96}$$

where

$$\zeta(p+1) = \sum_{n=1}^{\infty} n^{-(p+1)} \tag{E-97}$$

and

$$F_p\left(\frac{\omega_i}{N}\right) = \int_0^{\cos^{-1}\left(\frac{\omega_i}{N}\right)} \tan^p \theta \cdot d\theta \quad . \tag{E-98}$$

With these normalization factors, the temporal frequency energy spectrum becomes

$$E(\omega) = \frac{E}{2\,F_p\left(\frac{\omega_i}{N}\right)}\;\frac{[N^2 - \omega^2]^{\frac{p-1}{2}}}{\omega^p} \quad . \tag{E-99}$$

The temporal spectra for vertical velocity, horizontal velocities, and vertical displacement are obtained from (E-87) - (E-89). Thus, for the displacement one obtains

$$\Phi_{\eta\eta}(0,y,\ ) = \left(\frac{E}{DN^2\rho_c F_p\left(\frac{\omega_i}{N}\right)\zeta(p+1)}\right)\frac{[N^2 - \omega^2]^{\frac{p-1}{2}}}{\omega^p}\;\sum_{n=1}^{\infty}\frac{\sin^2\frac{n\pi y}{D}}{n^{p+1}} , \tag{E-100}$$

which has the same form as the energy spectrum. The vertical velocity spectrum differs from (E-100) by the factor ω^2:

$$\Phi_{yy}(0,y,\omega) = \omega^2\;\Phi_{\eta\eta}(0,y,\omega) \quad . \tag{E-101}$$

Also,

$$\Phi_{xx}(0,y,\omega) = \Phi_{zz}(0,y,\omega) = \frac{E}{2DN^2\,F_p\left(\frac{\omega_i}{N}\right)\zeta(p+1)}\;\frac{[N^2-\omega^2]^{\frac{p+1}{2}}}{\omega^p}\;\sum_{n=1}^{\infty}\frac{\cos^2\frac{n\pi y}{D}}{n^{p+1}} \quad . \tag{E-102}$$

These equations are based on the rather artificial model of a constant Brunt-Väisälä frequency. There are no caustics (turning points) and all modes from the lowest to the highest are affected by the boundary condition on the ocean bottom.

It is perhaps worth comparing the preceding results with those of Garrett and Munk [8]. In arriving at their formulation they do not employ our assumption on excitation, viz., Eq. (E-73). Also, they do not use a constant profile but rather one which is exponentially decreasing. They consider only

higher order modes with caustic boundaries (i.e., the ocean is assumed so deep that only the few lowest order modes, which they neglect, are affected by the ocean bottom). Although such features have no exact counterparts in a constant N-profile, in the frequency range $\omega << N$ we would expect some agreement with the Garrett and Munk spectrum. Indeed, the Garrett and Munk analysis does not apply to frequencies close to the local $N(y)$ (i.e., one must have $\omega << N(y)$ for all y). If one sets ω_i to zero in the Garrett and Munk formula for $E(\omega)$, the energy spectrum behaves as $1/\omega^2$. This agrees with the functional form obtained in (E-99) for $\omega << N$ and p = 2.

This behavior of energy with frequency in fact follows from the general formula (E-86) subject to (E-90) and p = 2 provided we employ the WKB approximation for all modes. The wave-number frequency relations under the WKB approximation are

$$K_n(\omega) \sim \frac{n\pi\omega}{\int_{-D}^{0} [N^2-\omega^2]^{\frac{1}{2}}\, dy} , \tag{E-103a}$$

and

$$v_{gn}(\omega) = \frac{\left[\int_{-D}^{0} [N^2-\omega^2]^{\frac{1}{2}}\, dy\right]^2}{n\pi \int_{-D}^{0} N^2[N^2-\omega^2]^{-\frac{1}{2}}\, dy} . \tag{E-103b}$$

Consequently,

$$K_n v_{gn} = \frac{\int_{-D}^{0} [N^2-\omega^2]^{\frac{1}{2}}\, dy}{\int_{-D}^{0} N^2[N^2-\omega^2]^{-\frac{1}{2}}\, dy} , \tag{E-103c}$$

which is independent of n. If we substitute these expressions in

(E-86) with $I(\underline{K}) = CK^{-p}$, the energy spectrum assumes the form

$$E(\omega) \sim \frac{C\rho_o}{\pi^p} \frac{1}{\omega^p} \left[\int_{-D}^{0} (N^2-\omega^2)^{\frac{1}{2}} \, dy \right]^p \int_{-D}^{0} N^2(N^2-\omega^2)^{-\frac{1}{2}} dy \sum_n \frac{1}{n^{p+1}} \cdot \quad \text{(E-104)}$$

Under the assumption that $\omega \ll N(y)$, this reduces to

$$E(\omega) \sim \frac{C\rho_o}{\pi^p} \left[\int_{-D}^{0} N dy \right]^{p+1} \sum_n \frac{1}{n^{p+1}} \cdot \frac{1}{\omega^p} , \quad \text{(E-105)}$$

and with $p = 2$ one again recovers the functional form of the Garrett and Munk energy spectrum. This compatibility with the Garrett and Munk spectrum applies to $E(\omega)$, but not necessarily to the temporal spectra of the velocities and displacement, viz., Eqs. (87) - (89), since these spectra contain an explicit dependence on y through the eigenfunctions. Garrett and Munk, on the other hand, integrate over the height coordinate to eliminate what they term "fine structure fluctuations", i.e., the explicit dependence on y. From the standpoint of a statistical description based on the theory of spatially homogeneous and stationary processes, there appears to be no basis for such an "averaging" operation.

D. INTERNAL WAVE SPECTRA FOR AN EXPONENTIAL VÄISÄLÄ FREQUENCY PROFILE

We now consider the exponential N(y) profile employed by Garrett and Munk in constructing their energy spectra. Instead of initially assuming a finitely deep ocean and then going over to the deep ocean in the limit, we shall assume an infinitely deep ocean at the outset. Thus, for

$$N(y) = N(0) \exp (y/b); \; -\infty < y < 0 \quad , \qquad \text{(E-106)}$$

the eigenvalue problem can be solved exactly since the equation

can be transformed into the Bessel equation. When normalized in accordance with (E-65) the eigenfunctions assume the form

$$\phi_n(y) = \frac{1}{N(0)}\sqrt{\frac{2}{b}}\ \frac{1}{\dot{J}_{Kb}(X_{n;Kb})}\ J_{Kb}(X_{n;Kb}\ e^{y/b}) \quad , \tag{E-107}$$

where

$$J_{Kb}(X_{n;Kb}) = 0 \quad . \tag{E-108}$$

The $\phi_n(y)$ satisfy the standard rigid lid boundary condition at $y = 0$, i.e., $\phi_n(o) = 0$; the second boundary condition is $\lim \phi_n(y) \to o$ as $y \to -\infty$. It should be apparent that these eigenfunctions cannot approach those for a constant N profile, i.e., $b \to \infty$, since the lower boundary of all modes comprises caustic surfaces.

The dispersion relation is given by

$$\frac{\omega}{N(0)} = \frac{Kb}{X_{n;Kb}} \quad . \tag{E-109}$$

With the aid of the differentiation formula for Bessel function with respect to order* one finds for the group speed

$$Kv_{gn}(\omega) = \omega\left[1-2\left(\frac{\omega}{N(0)}\right)^2 \frac{\displaystyle\int_0^{X_{n;Kb}} J^2_{Kb}(t)\ \frac{dt}{t}}{J^2_{Kb+1}(X_{n;Kb})}\right] \quad . \tag{E-110}$$

*The formula in question is

$$\frac{dj}{d\nu} = \frac{2\nu}{j\ J^2_{\nu+1}(j)} \int_0^j J^2_\nu(t)\ \frac{dt}{t} \quad , \text{ with } J_\nu(j) = 0$$

(G.N. Watson, "Theory of Bessel Functions", Cambridge 1958, p. 508.)

We now consider some limiting forms. First assume that $Kb >> 1$. A typical value for b is 1300 meters. If the longest wavelength of interest is about 200 meters, then $Kb \approx 40$ which is compatible with $Kb >> 1$ for all shorter wavelengths. The zeroes of the Bessel function in (D-108) lie in the range $X_{n;Kb} > Kb$. For large Kb and

$$\left| e^{y/b} X_{n;Kb} - Kb \right| \leq O(Kb)^{\frac{1}{3}} \quad , \tag{E-111}$$

the Bessel function may be approximated by the Airy function Ai, viz.,

$$J_{Kb}(e^{y/b} X_{n;Kb}) \sim \frac{2^{\frac{1}{3}}}{2(Kb)^{\frac{1}{3}}} \; Ai\left[-2^{\frac{1}{3}} \tau_n(y)\right] , \tag{E-112}$$

where

$$e^{y/b} X_{n;Kb} = Kb + (Kb)^{\frac{1}{3}} \tau_n(y) \quad . \tag{E-113}$$

If we denote by σ_n the nth zero of $Ai(-\sigma)$, then $\sigma_n = \tau_n(0) 2^{\frac{1}{3}}$ and

$$X_{n;Kb} = Kb + \left(\frac{Kb}{2}\right)^{\frac{1}{3}} \sigma_n ; \; n = 1,2, \; .... \tag{E-114}$$

The dispersion relation (E-109) may now be written

$$\frac{\omega}{N(0)} = \frac{Kb}{Kb + \left(\frac{Kb}{2}\right)^{\frac{1}{3}} \sigma_n} = \frac{1}{1 + \frac{1}{2} \sigma_n \left(\frac{2}{Kb}\right)^{\frac{2}{3}}} \quad . \tag{E-115}$$

For higher order modes, i.e.,

$$\left| X_{n;Kb} \; e^{y/b} - Kb \right| > O \; (Kb)^{\frac{1}{3}} \quad ,$$

Eq. (E-115) no longer applies and one must employ the Debye formulae (WKB approximation). In that case the dispersion relationship is given implicitly by

$$K_n(\omega) = \frac{\left(n-\frac{1}{4}\right)\pi\omega}{\int_{y_1}^{0} \sqrt{N^2(y)-\omega^2}\; dy} , \qquad \text{(E-116)}$$

with the turning point y_1 defined by

$$\omega = N(0) \exp\, (y_1/b) \quad .$$

If we restrict ourselves to the range of Kb encompassed by (E-111), then (E-115) yields*

$$\frac{Kb}{2} = \left(\frac{\sigma_n}{2}\right)^{\frac{3}{2}} \left[\frac{\frac{\omega}{N(0)}}{1-\frac{\omega}{N(0)}}\right]^{\frac{3}{2}} \quad . \qquad \text{(E-117)}$$

Note that for large Kb and moderate σ_n, ω must be close to N(0) since by the assumption in (E-114) $\sigma_n = 0(1)$. To this extent the situation is no different than in the former case of a constant profile, Eq. (E-91). By differentiating (E-117) with respect to K, one finds

$$Kv_{gn}(\omega) = \frac{2}{3}\, N(0) \left[\frac{\omega}{N(0)}\right]^{-3} \left[1 - \frac{\omega}{N(0)}\right] \quad .$$

---

*More generally for large Kb,

$$Kb = \frac{2}{3}\, \sigma_n^{\frac{3}{2}} \quad \frac{\eta}{\sqrt{1-\eta^2} - \eta\cos^{-1}\eta} \quad ; \ \eta = \frac{\omega}{N(0)} \; .$$

For $\eta \ll 1$, $\sigma_n \gg 1$, $\frac{2}{3}\sigma_n^{\frac{3}{2}} \cong (n - \frac{1}{4})\,\pi$ the result reduces to WKB approximation for the exponential profile; for $\eta \approx 1$, one obtains (E-117).

Note that the product of K and mode group speed is independent of n, as was also the case under the WKB approximation. For the energy frequency spectrum one obtains the following asymptotic result:

$$E(\omega) \sim 3\pi\rho_o \, C \left(\frac{b}{2}\right)^{p+1} N(0) \left[\frac{\omega}{N(0)}\right]^{\frac{7-3p}{2}} \left[1 - \frac{\omega}{N(0)}\right]^{\frac{3p+1}{2}} . \qquad \text{(E-118)}$$

In this case $E(\omega) \to 0$ as $\omega \to N(0)$ for all $p > 1$. This shows that under the assumption that the mode excitation obeys Milder's energy partitioning hypothesis, the energy-spectrum vanishes at the maximum Väisälä frequency even if the lower boundary is formed entirely of caustics.

Garrett and Munk employ the asymptotic forms of the Bessel functions for large arguments and thus consider the range of high mode numbers only. This corresponds to the other extreme of the frequency scale, viz., $\omega \ll N(0)$, within which (E-111) does not hold. For large arguments

$$J_{Kb}(X_{n;Kb}) \sim 2\sqrt{\frac{2}{\pi X_{n;Kb}}} \cos(X_{n;Kb} - \pi/4 - Kb\,\pi/2)$$

and

$$X_{n;Kb} - \pi/4 - Kb\,\frac{\pi}{2} \cong (2n-1)\,\pi/2 ,$$

or

$$X_{n;Kb} \cong \left[n - \frac{1}{4}\right] \pi + Kb\,\pi/2 .$$

Since the preceding asymptotic form holds only if $X_{n;Kb} \gg Kb$,

$$X_{n;Kb} \approx (n - \tfrac{1}{4})\,\pi ,$$

where n is large. Hence,

$$\omega \approx \frac{Kb\ N(0)}{(n-\frac{1}{4})\ \pi}\ , \quad Kv_{gn}\ (\omega) \approx \omega\ . \tag{E-119}$$

The temporal frequency energy spectrum now becomes

$$E(\omega) \approx 2C\rho_o \left(\frac{N(0)b}{\pi}\right)^{p+1} \left(\frac{1}{\omega^p}\right) \sum_n \frac{1}{\left[n-\frac{1}{4}\right]^{p+1}}\ . \tag{E-120}$$

The sum in (E-120) is to be extended only over large n. Equivalently (E-120) is an approximate representation for ω well below N (0). For p = 2 one again obtains the functional dependence on ω deduced by Garrett and Munk.

We now examine the energy density spectrum in wave number space, which is given by Eq. (E-78). When specialized to any deep ocean profile (i.e., lower boundary condition lim $\varphi_n \to 0$, $y \to -\infty$) in conjunction with Eq. (E-90), the energy density assumes the form

$$\varepsilon(K) = 2\pi\rho_o\ C\ K^{-p-2} \int_{-\infty}^{0} \left(1 - e^{2Ky}\right) N^2(y)\ dy\ . \tag{E-121}$$

Formally, $0 \leq K \leq \infty$. For $1 \leq p \leq 2$, $K\varepsilon(K)$ is not integrable over the full range of wave numbers. This can be remedied by either assuming a different functional dependence of I(K) on K near K = 0, or alternatively, by truncating $\varepsilon(K)$ below some lower wave number $K = K_c$. We choose the second alternative, as do Garrett and Munk in their paper. If we now specialize Eq. (E-121) to the exponential profile we obtain

$$\varepsilon(K) = \begin{cases} \pi\rho_o\, C_p\, N^2(0) b^2 \dfrac{K^{-p-1}}{1 + Kb} \;; & K_c < K < \infty \;, \\ \\ 0 \;; & 0 < K < K_c \;, \end{cases} \qquad \text{(E-122)}$$

where we have employed the notation C_p for C, to indicate its dependence on p. With E the total energy per unit horizontal surface area,

$$E = \pi\rho_o b^2 C_p\, N^2(0) \int_{K_c}^{\infty} dK\, \frac{K^{-p}}{1 + Kb} \;, \qquad \text{(E-123)}$$

one finds

$$C_1 = \frac{E}{\pi\rho_o\, N^2(0)\, b^2 \ln \dfrac{1 + \nu_c}{\nu_c}} \;, \qquad \text{(E-124)}$$

$$C_2 = \frac{K_c b\, E}{\pi\rho_o\, N^2(0)\, b^3 \left[1 - \nu_c \ln \dfrac{1 + \nu_c}{\nu_c}\right]} \;, \qquad \text{(E-125)}$$

where

$$\nu_c = K_c b \;. \qquad \text{(E-126)}$$

We shall determine the constants C_1 and C_2 from the theory and oceanographic data presented by Garrett and Munk. Since their theoretical model is fundamentally different from the one employed herein, only a partial correspondence with their theory can be established.

From Garrett and Munk, p. 252, following their Eq. (6.23) we find $\hat{\alpha}_1 = .04$ cpkm. This is the wave number below which the

Garrett and Munk energy wave number spectrum is truncated. Translated to our notation*, $\hat{\alpha}_1 \equiv K_c/2\pi = .04\ \mathrm{km}^{-1}$. Again from Garrett and Munk, Fig. 1, p. 228, $\hat{b} = 1.3$ km which equals b in our notation. Hence, $\nu_c = K_c b = .3267$. The (integrated) energy per unit area, $\hat{\rho}\ \hat{E}$, equals $.382 \times 10^4$ joules/m^2 (Garrett and Munk, p. 252). We have denoted this quantity by E. Finally from Garrett and Munk, the maximum Väisälä frequency $\hat{N} = 3$ cph $- 8.333 \times 10^{-4}$ cps. In our notation this yields $N(0) = 2\pi\hat{N} = 5.236 \times 10^{-3}$ rad/sec. We then find from Eqs. (E-123) and (E-124)(note $\rho_o \cong 10^3$ kg/m^2):

$$C_2 = 1.2166 \times 10^{-5} \quad , \tag{E-127a}$$

$$C_1 = 1.8727 \times 10^{-2}\ \mathrm{m} \quad , \tag{E-127b}$$

where C_2 is dimensionless. As a check on these two numbers, let us determine C_1 and C_2 by comparing the temporal frequency energy spectrum, Eq. (E-94), with the corresponding expression of Garrett and Munk. From (E-96) one finds

$$C_p \cong 5.33 \times 10^{-7} \frac{\pi^{p+1}\ b^{2-p}}{\zeta(p+1)} \quad . \tag{E-128'}$$

We have $\zeta(3) = 1.20205$, $\zeta(2) = 1.6449$. Consequently,

$$C_2 \cong 1.3757 \times 10^{-5} \quad ,$$

$$C_1 \cong 4.16 \times 10^{-3}\ \mathrm{m} \quad .$$

In vies of the crudeness of the approximation, Eq. (E-94)(which holds for a constant N profile), the agreement between the values in C_2 obtained by these two alternative procedures is rather remarkable. The discrepancy between the two values of C_1 is attributable to the fact that the total energy in the wave number domain is not conserved in the Garrett and Munk formula

*We avoid the popular but meaningless "unit" cpkm (cycles per kilometer) since a "cycle" is dimensionless.

for $E(\omega)$ when p is changed from 2 to 1. In all our numerical calculations we shall employ Eq. (E-127). With these constants, together with b = 1300 meters, $\nu_o = .327$, $N(0) = 5.236 \times 10^{-3}$ rad/sec and $\rho_o = 10^3$ kg/m³ substituted in Eq. (E-122), we obtain

$$K\varepsilon(K) = \begin{cases} 1.77 \dfrac{1}{(1+\nu)\,\nu^2} \text{ joules/m} & ; \; p = 2 \; , \\ \\ 2.097 \dfrac{1}{(1+\nu)\,\nu} \text{ joules/m} & ; \; p = 1 \; , \end{cases} \tag{E-128}$$

where ν is the dimensionless quantity

$$\nu = Kb \quad , \tag{E-129'}$$

and $K\varepsilon(K)$ is identically zero for $\nu < .327$. Thus for $\nu >> 1$, the dominant effect of a higher value of p is a more rapid decay of energy content with increasing wave numbers, as one would expect. The functional forms of the energy spectra are, of course, not independent of the Väisälä frequency profile. For example, consider a profile with a mixed layer near the ocean surface:

$$N(y) = \begin{cases} 0 & ; \; -y_m < y \leq 0 \quad , \\ \\ N_m \, e^{(y+y_m)/b} & ; \; -\infty < y < -y_m \quad . \end{cases} \tag{E-129}$$

One then finds from Eq. (E-121) with p = 2

$$K\varepsilon(K) = \pi\rho_o \, Cb^4 \, N_m^2 \; \frac{1 + \nu - e^{-2\nu \, y_m/b}}{\nu^3(1+\nu)} \quad . \tag{E-130}$$

Comparing this with the first equation in Eq. (E-128) we observe that although the functional behavior for small and intermediate ν differs, the decay law for large ν is the same in both cases, viz., ν^{-3}.

The exact form of the temporal frequency energy spectrum is more complicated. Employing (E-86) together with $I(K) = CK^{-p}$ and the relations for the exponential profile in (E-109) and (E-110), one obtains the following:

$$E(\omega) = 8.792 \times 10^5 \sum_{n=1}^{\infty} \eta \frac{\nu_n^{-3}}{1 - 2\eta^2 \dfrac{\int_0^{\nu_n/\eta} J_{\nu_n}^2(t)\,\dfrac{dt}{t}}{J_{\nu_n+1}^2\left(\nu_n/\eta\right)}} \text{ joules/m}^2\text{ Hz}, \tag{E-131}$$

for $p = 2$, and

$$E(\omega) = 10.41 \times 10^5 \sum_{n=1}^{\infty} \eta \frac{\nu_n^{-2}}{1 - 2\eta^2 \dfrac{\int_0^{\nu_n/\eta} J_{\nu_n}^2(t)\,\dfrac{dt}{t}}{J_{\nu_n+1}^2\left(\nu_n/\eta\right)}} \text{ joules/m}^2\text{ Hz}, \tag{E-132}$$

for $p = 1$, where

$$\eta = \frac{\omega}{N(0)} \quad , \tag{E-133}$$

and $\nu_n = \nu_n(\eta)$ is the solution of $\eta = \dfrac{\nu}{X_{n;\nu}(\eta)}$ for ν. The physical constants employed in (E-131) and (E-132) are the same as given in the preceding discussion. The results of a numerical evaluation of $E(\omega)$ are plotted in Fig. E-1. The series were found to converge quite rapidly, so that over most of the range only a few modes were required.

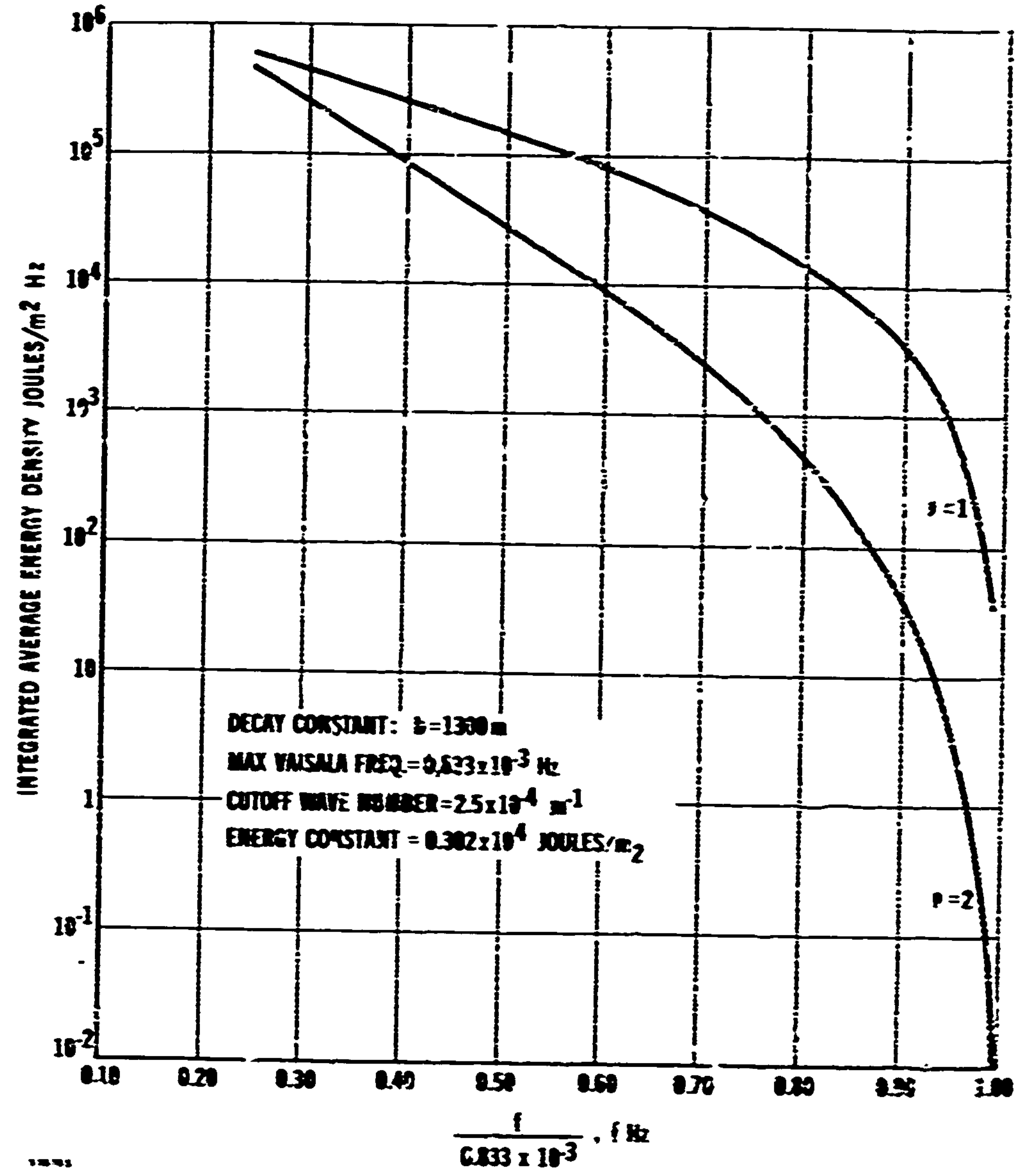

FIGURE E-1. Integrated average energy density of internal waves in an exponentially stratified ocean.

E. TOWED SPECTRA

The temporal spectra of displacement and internal wave velocities discussed in (a) presuppose that the observation platform is perfectly stationary with respect to earth-fixed coordinates. Actual internal wave measurements almost invariably involve moving (or towed) sensors. We, therefore, would like to formulate expressions for temporal spectra that would be observed from a moving measurement platform. We suppose that the sensor is moving at a uniform velocity $\underline{V}$ at depth y. Let the orientation of $\underline{V}$ with respect to x-axis be α, as shown in Fig. E-2.

Milder's mode partitioning hypothesis as given by Eq. (E-73) will be employed throughout. However, initially, we shall leave the functional form of the excitation function $I(\underline{K})$ unspecified. The auto-correlation function of the vertical fluid velocity as would be observed on the platform moving with a velocity $\underline{V}$ is

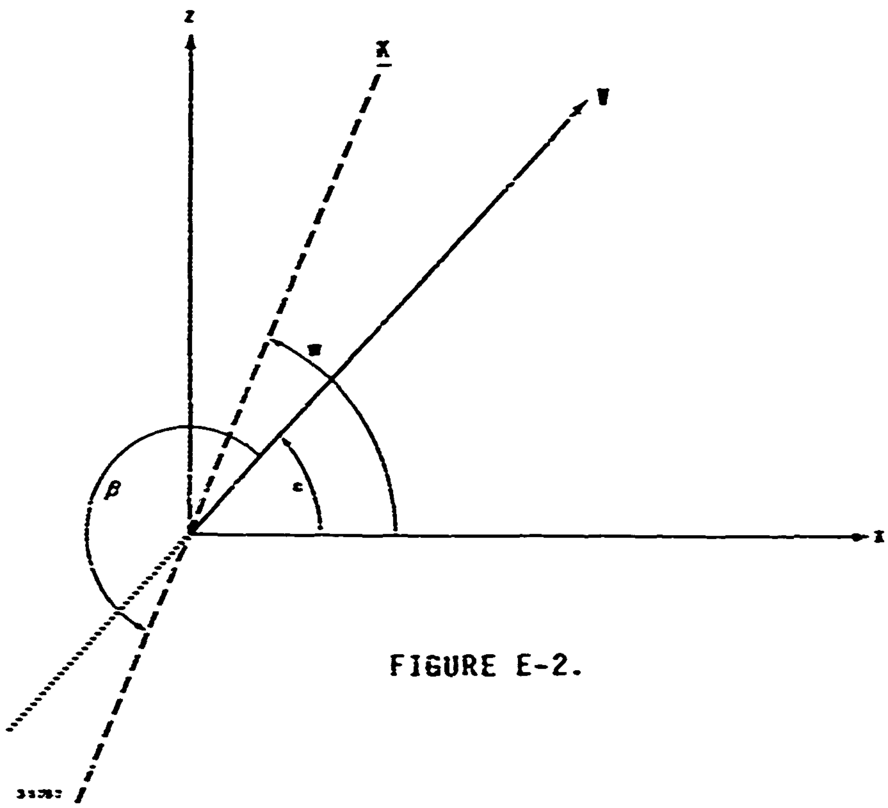

FIGURE E-2.

given by Eq. (E-57) with $\underline{\rho}$ replaced by $\underline{V}\,\tau$:

$$R_{yy}^{(V)}(y,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d^2\underline{K}\, \frac{e^{-i\underline{K}\cdot\underline{V}\tau}}{K^3} \sum_n \phi_n^2(y)\, \Omega_n^4 \left[I(\underline{K})\, e^{i\Omega_n\tau} + I(-\underline{K})\, e^{-i\Omega_n\tau} \right] \quad . \qquad \text{(E-134a)}$$

The correlation function of the vertical particle displacement is obtained from the preceding expression by dividing each term in the series by Ω_n^2 :

$$R_{\eta\eta}^{(V)}(y,\tau) = \frac{1}{2\pi}\iint_{\infty}^{\infty} d^2\underline{K}\, \frac{e^{-i\underline{K}\cdot\underline{V}\tau}}{K^3} \sum_n \phi_n^2(y)\, \Omega_n^2 \left[I(\underline{K})\, e^{i\Omega_n\tau} + I(-\underline{K})\, e^{-i\Omega_n\tau} \right] \quad , \qquad \text{(E-134b)}$$

The superscript V is employed to distinguish these correlation functions from those in a stationary reference frame. The temporal spectra are defined as Fourier transforms of these correlation functions. Thus, upon changing to polar coordinates, the spectrum of the vertical fluid velocity is

$$\Phi_{yy}^{(V)}(y,\omega) = \int_0^{2\pi} d\kappa \int_0^{\infty} \frac{dK}{K^2}$$

$$\cdot \sum_n \phi_n^2\, \Omega_n^4 \cdot \left\{ I(K,\kappa)\, \delta\left[\Omega_n - \underline{K}\cdot\underline{V} - \omega\right] + I(K,\kappa+\pi)\, \delta\left[\Omega_n + \underline{K}\cdot\underline{V} + \omega\right] \right\} \quad , \qquad \text{(E-135)}$$

where

$$\underline{K}\cdot\underline{V} = KV\cos(\kappa - \alpha) \quad . \qquad \text{(E-136)}$$

The spectrum of the vertical displacement $\Phi_{\eta\eta}^{(V)}(y,\omega)$ is given by the same expression, provided only that Ω_n^4 is replaced by Ω_n^2. One integration (either over K or κ) can be carried out by simply evaluating the integrand at the points for which the arguments of the delta functions vanish.

Formally, the procedure is trivial. Nevertheless, care must be exercised so that all contributions are correctly accounted for. This "bookkeeping" is facilitated if one rewrites Eq. (E-135) in a slightly different from. For this purpose we first change the variable of integration from ψ to β with

$$\beta = \psi - \alpha + \pi \quad . \tag{E-137}$$

The geometrical relationship among the three angles is depicted in Fig. E-2. We shall also stipulate at the outset that $\omega \geq 0$, $\Omega_n(K) > 0$, $KV > 0$. Since the integrand in Eq. (E-135) can always be taken as periodic in ω with period 2π, we can take the limits of integration in β to run from $-\pi$ to $+\pi$. The portion of the integral over $-\pi$ to 0 can then be transformed into one from 0 to π, so that Eq. (E-135) becomes

$$\Phi^{(V)}_{yy}(y,\omega) = \int_0^\pi d\beta \int_0^\infty \frac{dK}{K^2}$$

$$\cdot \sum_n \phi_n^2 \, \Omega_n^4 \left\{ J(K, \alpha+\pi, \beta)\, \delta\left[\Omega_n + KV \cos\beta - \omega \right] + J(K,\alpha,\beta)\, \delta\left[\Omega_n - KV \cos\beta - \omega \right] \right\} , \tag{E-138}$$

where

$$J(K,\alpha,\beta) = I(K,\alpha+\beta) + I(K,\alpha-\beta) \quad . \tag{E-139}$$

We will first integrate over K. Consider the first of the two delta functions in Eq. (E-138). There will be contributions whenever

$$\Omega_n(K) + KV \cos\beta - \omega = 0 \quad , \tag{E-140}$$

or

$$K = \frac{\omega - \Omega_n(K)}{V \cos \beta} \quad .$$

The principal interval in β is $-1 \leq \cos \beta \leq 1$ $(0 < \beta < \pi)$. Evidently the solution of Eq. (E-140) for K is a function of ω and $V \cos \beta$, which we denote by $K_n(\omega, V \cos \beta)$. An alternative form of Eq. (E-140) is therefore

$$\omega = \Omega_n[K_n(\omega, V \cos \beta)] + VK_n(\omega, V \cos \beta) \cos \beta \quad . \qquad \text{(E-141)}$$

For any fixed ω, the possible solutions of Eq. (E-140) for K as a function of β can be exhibited with the aid of a graphical construction by plotting $\Omega_n(K) + KV \cos \beta$ as a function of K with β as a parameter. The roots of Eq. (E-140) are then determined from the intersections of these parametric curves with a line ω = constant drawn parallel to the abscissa. A typical dispersion curve $\Omega_n(K)$ is a monotonically increasing function bounded from above by the asymptote N_{max}, the maximum Väisälä frequency. If $0 \leq \beta < \pi/2$, a typical plot of $\Omega_n(K) + KV \cos \beta$ will also represent a monotonically increasing function. On the other hand, when $\pi/2 < \beta \leq \pi$ the curves will have a peak, corresponding to a value of K given by the solution of

$$v_{gn}(K) \equiv \frac{d\Omega_n(K)}{dK} = -V \cos \beta \quad , \qquad \text{(E-142)}$$

i.e., at a value of K such that the group speed of the wave equals the component of the platform velocity along the propagation direction of the internal wave. If

$$\left. \frac{d\Omega_n(K)}{dK} \right|_{K=0} > V \quad , \qquad \text{(E-143)}$$

then by virtue of the monotonicity of $\Omega_n(K)$, Eq. (E-142) has always a solution* for $\beta = \pi$. This means that there is a wave number K such that the corresponding wave group speed equals the platform velocity. Since typically internal wave group speeds are on the order of a fraction of a meter/sec. this condition can be satisfied only for very small platform velocities. For faster platform velocities, viz., if

$$\left.\frac{d\Omega_n(K)}{dK}\right|_{K=0} < V \quad , \tag{E-144}$$

Eq. (E-142) can have a solution only for β strictly less than π. Typical plots of $\Omega_n(K) + KV \cos \beta$ are shown in Fig. E-3 for the case of slow platform speeds, viz., when Eq. (E-143) holds.

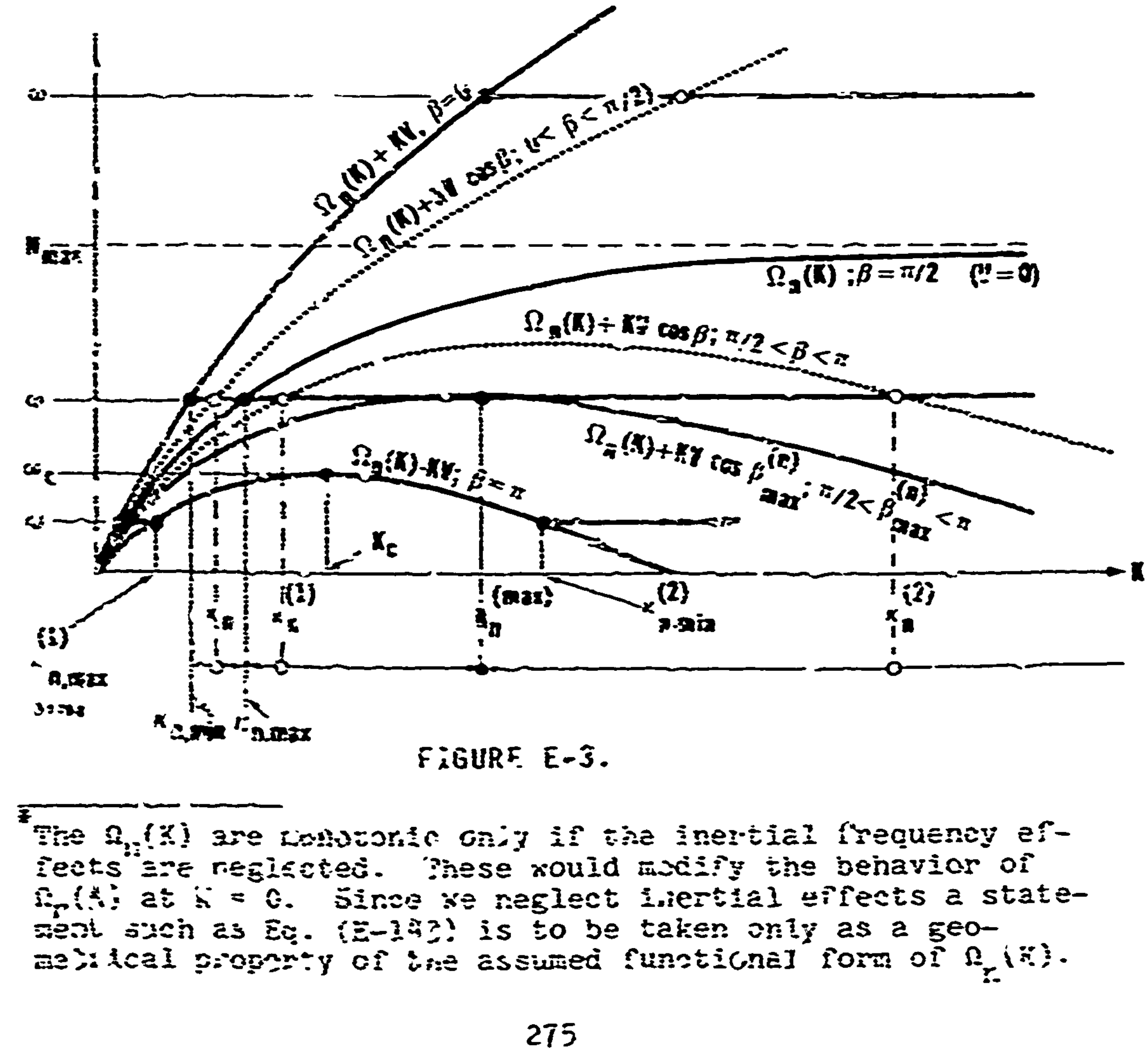

FIGURE E-3.

*The $\Omega_n(K)$ are monotonic only if the inertial frequency effects are neglected. These would modify the behavior of $\Omega_n(K)$ at $K = 0$. Since we neglect inertial effects a statement such as Eq. (E-142) is to be taken only as a geometrical property of the assumed functional form of $\Omega_n(K)$.

Consider first the case $\omega_c < \omega < N_{max}$ where ω_c corresponds to the maximum ordinate of $\Omega_n(K) - KV$. Thus,

$$\omega_c = \Omega_n(K_c) - K_c V \quad , \tag{E-145}$$

with K_c determined from Eq. (E-142) with $\beta = \pi$. From Fig. E-3 we observe that for $0 < \beta < \pi/2$ there is only one root, which we denote by $\kappa_n(\omega,\beta)$. In the figure its typical location is represented by an open circle, while the two extreme positions (corresponding to $\beta = 0$ and $\beta = \pi/2$) are represented by filled circles. Thus, $\kappa_{n,min} \leq \kappa_n \leq \kappa_{n,max}$ with $\kappa_{n,max}$, $\kappa_{n,min}$ satisfying

$$\Omega_n(\kappa_{n,min}) + \kappa_{n,min} V = \omega \quad ,$$

$$\Omega_n(\kappa_{n,max}) = \omega \quad , \tag{E-146}$$

represents the range of spatial wave numbers contributing at this frequency.

For $\beta > \pi/2$ there are two roots: $\kappa_n^{(1)}$ and $\kappa_n^{(2)}$ corresponding to two possible intersections of the line ω = constant with the curve $\Omega_n(K) + KV \cos \beta$, $\pi/2 < \beta < \beta_{max}^{(n)}$. No real solutions are obtained for $\beta > \beta_{max}^{(n)}$ (see Fig. E-3). For $\omega_c < \omega < N_{max}$, $\beta_{max}^{(n)}$ is always less than π. As β approaches $\beta_{max}^{(n)}$, the two roots degenerate into a single root. The range of wave numbers contributed by $\kappa_n^{(1)}$ is comprised within the finite segment

$$\kappa_{n,max} \leq \kappa_n^{(1)} \leq \kappa_n^{(max)} \quad . \tag{E-147}$$

On the other hand, the range of contributing wave numbers associated with $\kappa_n^{(2)}$ is infinite, viz.,

$$\kappa_n^{(max)} \leq \kappa_n^{(2)} < \infty \quad . \tag{E-148}$$

Thus, with $\beta = \pi/2$, $\kappa_n^{(1)} = \kappa_{n,max}$, while $\kappa_n^{(2)}$ is at infinity. As β is increased, the first root increases, while the second root decreases from its initial position at infinity, both roots reaching their final common value K_n^{max} at $\beta = \beta_{max}^{(n)}$.

Suppose we now consider the frequency range $0 < \omega < \omega_c$. For $0 < \beta < \pi/2$, the situation is the same as before, i.e., we have the single root κ_n. When $\beta > \pi/2$, we again obtain two roots: $\kappa_n^{(1)}$ and $\kappa_n^{(2)}$. Now, however, the maximum value of β is equal to π, so that $\kappa_{n,max}^{(1)} \neq \kappa_{n,min}^{(2)}$, i.e., as β increases from $\pi/2$ toward its maximum value of π, the two roots remain distinct. If the platform velocity is reduced to zero, then $\kappa_n^{(2)} \to \infty$ while $\kappa_n^{(1)} \to \kappa_n$, which then is determined by the simple dispersion relation $\Omega_n(\kappa_n) = \omega$.

In the frequency range $\omega > N_{max}$ the situation is substantially simpler. Since the maximum of $\Omega_n(K) + K\cos\beta$ for $\pi/2 < \beta < \pi$ never exceeds N_{max}, there are no roots for $\beta > \pi/2$, and only the single root κ_n in the range $0 \leq \beta \leq \pi/2$.

The curves in Fig. E-3 have been drawn for the case of low platform velocities, i.e., in the sense of Eq. (E-143). Clearly, for fast platform velocities, i.e., when (E-144) holds, the preceding discussion for $\omega_c < \omega < N_{max}$ applies without modification, except that $\omega_c = 0$, since now the curve $\Omega_n(K) - KV$ in Fig. E-3 lies outside the range of positive ordinates. Consequently, the range of β is in all cases $0 < \beta < \beta_{max}^{(n)} < \pi$. For $\omega > N_{max}$, again only one root is obtained, viz., κ_n for $0 \leq \beta \leq \pi/2$.

We must still consider the roots of the argument of the second delta function in (E-138), i.e.,

$$\omega = KV\cos\beta - \Omega_n(K). \qquad \text{(E-149)}$$

Since $\omega > 0$, only the range $0 < \beta < \pi/2$ is of interest. For any positive monotonically increasing function $\Omega_n(K)$ such as in Fig. E-3, (E-149) can have only one root, which we denote by $\kappa_n^{(3)}(\omega,\beta)$.

The range of wave numbers encompassed by this root is infinite,

$$\kappa^{(3)}_{n,min} < \kappa^{(3)}_{n}(\omega,\beta) < \infty, \qquad \text{(E-150)}$$

where the infinite endpoint corresponds to $\beta = \pi/2$. We also note that

$$\lim_{V\to 0} \kappa^{(3)}_{n,min} \to \infty,$$

so that this root, just like $\kappa^{(2)}_{n}$ arising from the first delta function, describes a purely motion induced effect.

By way of summary we list in detail the various parameter ranges in the "augmented" dispersion relationships (E-140) and (E-149):

$\omega > N_{max}$

(i) $\Omega_n(K) + KV\cos\beta - \omega = 0$

has one solution $K = \kappa_n(\omega,\beta)$; $\kappa_{n,min} < \kappa_n < \infty$; $0 \leq \beta \leq \pi/2$.

(ii) $\Omega_n(K) - KV\cos\beta + \omega = 0$

has one solution $K = \kappa^{(3)}_{n}(\omega,\beta)$; $\kappa^{(3)}_{n,min} < \kappa^{(3)}_{n} < \infty$; $0 \leq \beta \leq \pi/2$.

(E-151)

$\omega_c < \omega < N_{max}$

(i) $\Omega_n(K) + KV\cos\beta - \omega = 0$

has three solutions, 1) $K=\kappa_n(\omega,\beta)$, $\kappa_{n,min}<\kappa_n<\kappa_{n,max}, 0\leq\beta\leq\pi/2$

2) $K=\kappa_n^{(1)}(\omega,\beta)$, $\kappa_{n,max}<\kappa_n^{(1)}<K_n^{(max)}, \pi/2\leq\beta\leq\beta_{max}^{(n)}$

3) $K=\kappa_n^{(2)}(\omega,\beta)$, $K_n^{(max)}<\kappa_n^{(2)}<\infty$, $\pi/2\leq\beta\leq\beta_{max}^{(n)}$.

(ii) $\Omega_n(K) - KV\cos\beta + \omega = 0$

has one solution $K = \kappa_n^{(3)}(\omega,\beta)$, $\kappa_{n,min}^{(3)} < \kappa_n^{(3)} < \infty$, $0 \leq \beta \leq \pi/2$.

(E-152)

$0 < \omega < \omega_c$

The same as in the preceding case, except that $\beta_{max}^{(n)} = \pi$ and $\kappa_{n,max} < \kappa_n^{(1)} < \kappa_{n,max}^{(1)}$, $\kappa_{n,min}^{(2)} < \kappa_n^{(2)} < \infty$, where

$$\kappa_{n,max}^{(1)} \neq \kappa_{n,min}^{(2)}. \tag{E-153}$$

With the aid of these results we can now integrate (E-138) with respect to K. For the case $\omega > N_{max}$ we obtain

$$\Phi_{yy}^{(V)}(y,\omega) = \int_0^{\pi/2} d\beta \left\{ \sum_n \frac{j(\kappa_n,\alpha+\pi,\beta)\phi_n^2(y,\kappa_n)\Omega_n^4(\kappa_n)}{[v_{gn}(\kappa_n)+V\cos\beta][\kappa_n]^2} + \frac{j(\kappa_n^{(3)},\alpha,\beta)\phi_n^2(y,\kappa_n^{(3)})\Omega_n^4(\kappa_n^{(3)})}{[V\cos\beta - v_{gn}(\kappa_n^{(3)})][\kappa_n^{(3)}]^2} \right\}, \tag{E-154}$$

where we have used the notation $\phi_n(y,\kappa_n)$ to bring attention to the fact that the eigenfunctions generally depend on K which, in this evaluation, must be replaced by κ_n or $\kappa_n^{(3)}$. For the

case $\omega_c < \omega < N_{max}$ we have

$$\Phi_{yy}^{(V)}(y,\omega) = \int_0^{\pi/2} d\beta \left\{ \sum_n \frac{j(\kappa_n,\alpha+\pi,\beta)\phi_n^2(y,\kappa_n)\Omega_n^4(\kappa_n)}{[v_{gn}(\kappa_n)+V\cos\beta][\kappa_n]^2} + \frac{j(\kappa_n^{(3)},\alpha,\beta)\phi_n^2(y,\kappa_n^{(3)})\Omega_n^4(\kappa_n^{(3)})}{[V\cos\beta-v_{gn}(\kappa_n^{(3)})][\kappa_n^{(3)}]^2} \right\}$$

$$+\sum_{\ell=1}^{2}\sum_n \int_{\pi/2}^{\beta_{max}^{(n)}} d\beta \, \frac{j(\kappa_n^{(\ell)},\alpha+\pi,\beta)\phi_n^2(y,\kappa_n^{(\ell)})\Omega_n^4(\kappa_n^{(\ell)})}{|v_{gn}(\kappa_n^{(\ell)})+V\cos\beta|[\kappa_n^{(\ell)}]^2} \; . \qquad \text{(E-155)}$$

The expression valid within the frequency range $0 < \omega < \omega_c$ can be obtained from (E-155) by replacing $\beta_{max}^{(n)}$ by π. The range of integration is then the same for all modes, so that the integral can be brought outside the summation signs. An exact evaluation of these expressions would be a formidable numerical task primarily because of the explicit dependence on the "augmented" dispersion relations.

We now consider the special case of fast "tow speeds" V and frequencies greater than the maximum Väisälä frequency. Clearly for $\beta \neq \pi/2$ and V sufficiently large, both of the "augmented" dispersion relations, viz., (E-140) and (E-149) have identical asymptotic solutions for K, viz.,

$$K \sim \frac{\omega}{V\cos\beta} \; . \qquad \text{(E-156)}$$

If the excitation function j appearing in the numerator of (E-154) decays sufficiently rapidly in K space, then the net contribution from the neighborhood of $\beta \approx \pi/2$ is small (recall that both κ_n and $\kappa_n^{(3)} \to \infty$ as $\beta \to \pi/2$). Under these conditions we can set

$$\kappa_n \sim \frac{\omega}{V\cos\beta} = K, \ \kappa_n^{(3)} \sim \frac{\omega}{V\cos\beta} = K, \qquad \text{(E-157)}$$

and thus obtain an asymptotic approximation to (E-154) for large V. Since to the same order of approximation,

$$v_{gn}(\kappa_n) << V\cos\beta \ ,$$

$$v_{gn}(\kappa_n^{(3)}) << V\cos\beta \ ,$$

the asymptotic approximation to Eq. (E-154) becomes

$$\Phi_{yy}^{(V)}(y,\omega) \underset{V\sim\infty}{\sim} \frac{1}{\omega}\int_0^{\pi/2} d\beta[j(K,\alpha+\pi,\beta) + j(K,\alpha,\beta)] \frac{1}{K} \cdot \sum_n \phi_n^2(y)\Omega_n^4(K) \ , \qquad \text{(E-158)}$$

where K is given by (E-156). Eq. (E-158) holds for tow speeds which are much larger than the wave group velocities and for $\omega > N_{max}$. Unfortunately, a similar asymptotic development does not hold for $\omega < N_{max}$, i.e , in (E-155). This is clear from the dispersion curves in Fig. E-3, which show that $\beta = \pi/2$ need not correspond to a large wave number.

The closed form of the last sum we have already employed in the representation of the spatial spectrum. Thus, with a reference to (E-83), the preceding reads

$$\Phi_{yy}^{(V)}(y,\omega) \underset{V\sim\infty}{\sim} \frac{1}{\omega}\int_0^{\pi/2} d\beta K^3[j(K,\alpha+\pi,\beta) + j(K,\alpha,\beta)]\int_{-D}^{0} N^2(y'')g^2(y'',y)dy'' . \qquad \text{(E-159)}$$

The asymptotic expression for the spectrum of the displacement is obtained from (E-158) by replacing Ω_n^4 with Ω_n^2. The sum is then carried out with the aid of (F-5), with the result

$$\Phi_{\eta\eta}^{(V)}(y,\omega) \underset{V\sim\infty}{\sim} \frac{1}{2\omega}\int_0^{\pi/2} d\beta\,[(j(K,\alpha+\kappa,\beta) + j(K,\alpha,\beta)]\ f_D(K,y). \tag{E-160}$$

Just like the corresponding spatial spectra, (E-81) and (E-84), Eqs. (E-160) and (E-159) involve at most a dependence on the Väisälä profile, i.e., they do not depend explicitly on the details of the dispersion relations.

Suppose we assume that the excitation function $I(\underline{K})$ is isotropic in wave number space and of the form (E-90). If we also use the deep ocean approximation for $f_D(K,y)$, Eq. (E-77), then the asymptotic form of the (temporal) displacement spectrum in (E-160) becomes

$$\Phi_{\eta\eta}^{(V)}(y,\omega) \sim \frac{C}{\omega}\int_0^{\pi/2} \frac{d\beta}{K^p}\ (1-e^{2Ky}) \quad . \tag{E-161}$$

With the aid of the transformation $\xi = \sec\beta$ this integral can be put into the following form:

$$\Phi_{\eta\eta}^{(V)}(y,\omega) \underset{V\sim\infty}{\sim} CV^p\ \ \omega^{-p-1}\int_1^{\infty}\left(1-e^{2y\frac{\omega}{V}\xi}\right)\frac{d\xi}{\xi^{p+1}\sqrt{\xi^2-1}} \quad . \tag{E-162}$$

On the other hand, from (E-81) the spatial wave number spectrum of particle displacement for the same excitation function under the deep ocean assumption is

$$KS_{\eta\eta}(K) = \frac{C}{2\pi}K^{-p-1}\ (1-e^{2Ky}) \quad . \tag{E-163}$$

Comparing this with (E-162), we observe that the towed temporal spectrum contains as a factor the same characteristic decay law in frequency. This decay law is, however, modified by

the integral. Had we assumed a unidirectional wave number spectrum, then the dependence of $\Phi_{\eta\eta}^{(V)}$ on frequency would have matched exactly the functional dependence of the spatial spectrum on wave number. The discrepancy between the two functional forms is a measure of the effects of wave number isotropy. Such effects are weak only at high frequencies and large $K|y|$. Thus, since the frequency dependent part of the integral contains ω only in the argument of a decaying exponential (recall that $y < 0$, while $\xi > 1$), we have for $|\frac{\omega y}{V}| >> 1$,

$$\Phi_{\eta\eta}^{(V)}(y,\omega) \approx \left(CV^p \int_1^\infty \frac{d\xi}{\xi^{p+1}\sqrt{\xi^2-1}}\right)\omega^{-p-1} \quad . \qquad \text{(E-164)}$$

On the other hand, for $K|y| >> 1$, (E-163) gives

$$KS_{\eta\eta} \approx \frac{C}{2\pi} K^{-p-1} \quad . \qquad \text{(E-165)}$$

Of course, the form of (E-162) is sufficiently simple so that it can be evaluated for various combinations of tow speed and decay constant p. Since the towed spectrum is a measurable quantity, calculations based on (E-162) are subject to direct experimental verification. It is important to note that (E-162) holds for <u>any</u> Väisälä frequency profile in a sufficiently deep ocean (e.g., greater than 3000 m). The assumptions underlying (E-162) are: (1) Milder's energy partitioning hypothesis, (2) the functional dependence of the excitation function on wave number in the form CK^{-p}, (3) isotropy of the wave number spectrum, (4) fast tow speeds and $\omega > N_{max}$, and finally (5) the validity of the linearized theory together with stochastic stationarity in time and space. Of these assumptions, (2) and (3) are readily altered. Thus, expressions similar to (E-162) can be obtained for other than the power law functional dependence of the excitation function.

While the displacement spectrum is independent of the Väisälä frequency profile, the spectrum of the vertical velocity in (E-159) is profile dependent. For any specific profile (E-159) can be readily evaluated, again affording theoretical results that can be compared with experimental data.

APPENDIX F

TWO IDENTITIES INVOLVING SUMS OF WEIGHTED EIGENFUNCTION PRODUCTS

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We suppose that $\phi_n(y)$ is a complete (discrete) set of eigenfunctions to the eigenvalue problem

$$\left[\frac{d^2}{dy^2} + K^2\left(\frac{N^2(y)}{\Omega_n^2(K)} - 1\right)\right] \phi_n(y) = 0 \tag{F-1}$$

with the boundary conditions

$$\begin{aligned} &\phi_n(0) = 0 \ , \\ &\phi_n(-D) = 0 \quad ; \quad -D \le y \le 0 \ , \end{aligned} \tag{F-2}$$

or

$$\begin{aligned} &\phi_n(0) = 0 \ , \\ &\lim_{y \to -\infty} \phi_n(y) \to 0 \quad ; \quad -\infty \le y \le 0 \ . \end{aligned} \tag{F-3}$$

The eigenfunctions are normalized in accordance with

$$\int_{-D}^{0} N^2(y)\, \phi_n^2(y) \, dy = 1 \tag{F-4}$$

where $D \to \infty$ for the boundary conditions (F-3). Then the following identities hold:

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$$\sum_{n=1}^{\infty} \Omega_n^2(K)\ \phi_n(y)\ \phi_n(y') = -K^2 g(y,y') \quad , \tag{F-5}$$

$$\sum_{n=1}^{\infty} \Omega_n^4(K)\ \phi_n(y)\ \phi_n(y') = K^4 \int_{-D}^{0} N^2(y'')\ g(y'';y')\ g(y,y'')\ dy'' \tag{F-6}$$

where

$$g(y,y') = \begin{cases} \dfrac{\sinh K(y_< + D)\ \sinh Ky_>}{K \sinh KD} \ , \text{ for B·C (F-2)} & \text{(F-7)} \\ \\ \dfrac{e^{Ky_<}\ \sinh Ky_>}{K} \ , \text{ for B·C. (F-3)} & \text{(F-8)} \end{cases}$$

These identities can be inferred from a more general result given in Ref. [21]. Here we present a direct proof of these important relations.

<u>Proof</u>:

By virtue of the normalization (F-4), the "completeness" relation for the eigenfunctions may be written in the following form

$$\sum_{n=1}^{\infty} \phi_n(y)\ \phi_n(y') = \frac{\delta(y-y')}{N^2(y')} \tag{F-9}$$

Let $\mathcal{L} = \frac{d^2}{dy^2}$ and

$h = \sum_{n=1}^{\infty} \Omega_n^2\ \phi_n(y)\ \phi_n(y')$. Then upon taking account of (F-1)

one has

$$\mathcal{L}h = \sum_{n=1}^{\infty} \Omega_n^2 \, \phi_n(y')\mathcal{L}\phi_n(y) = \sum_{l=1}^{\infty} \Omega_n^2 \, \phi_n(y') \left(1 - \frac{N^2(y)}{\Omega_n^2}\right) K^2 \phi_n(y) \; .$$

In view of (F-9) this may be rewritten as follows:

$$\frac{d^2h}{dy^2} - K^2 h = - K^2 \; \delta(y-y') \quad .$$

With $h = - K^2 g$, g is the solution of

$$\left(\frac{d^2}{dy^2} - K^2\right) g = \delta(y-y') \quad . \tag{F-10}$$

Evidently h and g must satisfy the same boundary conditions as $\phi_n(y)$. Consequently, the Green's function problem in (F-10) must be solved subject to (F-2) or (F-3). One then finds that g is given by (F-7) and (F-8), respectively. This establishes (F-5).

To prove (F-6) we repeat the same procedure for the function

$$f(y,y') = \sum_{n=1}^{\infty} \Omega_n^* \;\; \phi_n(y) \; \phi_n(y')$$

Thus $\mathcal{L}f = \sum_{n=1}^{\infty} \Omega_n^* \, \phi_n(y')\mathcal{L}\phi_n(y) = \sum_{n=1}^{\infty} \Omega_n^* \, \phi_m(y') \left(1 - \frac{N^2(y)}{\Omega_n^2}\right) K^2\phi_n(y)$

which is equivalent to

$$\frac{d^2f}{dy^2} - K^2 f = - N^2(y) \; K^2 \sum_{n=1}^{\infty} \Omega_n^2 \; \phi_n(y) \; \phi_n(y')$$

$$= - N^2(y)\ K^2 h \quad = N^2(y)\ K^4\ g(y,y') \cdot$$

Thus, f(y,y') is the solution of

$$\left(\frac{d^2}{dy^2} - K^2\right)\ f(y,y') = K^4 N^2(y)\ g(y,y') \quad . \qquad \text{(F-11)}$$

Again, f(y,y') must satisfy the same boundary conditions as $\phi_n(y)$. Consequently, (F-11) is solved by the Green's function in (F-10), viz.,

$$f(y,y') = K^4 \int_{-D}^{0} dy''\ N^2(y'')\ g(y'',y')\ g(y,y'') \quad , \qquad \text{(F-12)}$$

which proves (F-6) .

Q.E.D.

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